Chapter 9 Differential Equations

9.1 Modeling with Differential Equations (page 586)

An *ordinary differential equation* (普通微分方程) is an equation that contains an unknown function and some of its derivatives. In a real world problem, we use the mathematical model in form of a differential equation because we often notice that changes occur and we want to predict future behavior on the basis of how current values change.

Models of Population Growth, page 586

Example 1 (page 586)**.** One model for the growth of a population is based on the assumption that the population grows at a rate proportional to the size of the population. That is a reasonable assumption for a population of bacteria or animals under ideal condition (unlimited environment, adequate nutrition, absence of predators, immunity from disease.)

Let's identify and name the variables in this model:

- t = time.
- $P = P(t)$ = the number of individuals in the population.

So we can get the differential equation:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP, \tag{1}$$

where k is the proportionality constant.

We know from Chapter 3 that exponential functions satisfy the equation (1). Here we can solve the equation by the method of integration:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP \Rightarrow$$

In this formula, we allow C to vary through all the real numbers, and we get the *family* of solution. But in real world problem, populations have only positive values and so we are interested only in the solutions with $C_0 > 0$. (We probably concerned only with values of t greater than the initial time $t = 0$.) We can put $t = 0$ and get $P(0) = C_0 e^k \cdot 0 = C_0$, so the constant C_0 turns out to be the initial population $P(0)$.

Example 2 (Logistic differential equation, page 587)**.** Example 1 shows a model for population growth under ideal conditions, but we have to recognize that a more realistic model must reflect that the fact that a given environment has limited resources. Many populations starts by increasing in an exponential manner, but the population levels off when it approaches its *carrying capacity* M (or decreases toward M if it ever exceed M, 最大負荷量). For a model to take into account both trends we make to assumptions:

- $\frac{\mathrm{d}P}{\mathrm{d}t} \approx kP$ if P is small. (Initially, the growth rate is proportional to P).
- $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$ if $P > M$. (P decreases if it ever exceeds M.)

A simple expression that incorporates both assumptions is given by the equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right). \tag{2}$$

Why this differential equation is reasonable?

- If P is small compared with M, then $\frac{P}{M}$ is closed to 0 and so $\frac{\mathrm{d}P}{\mathrm{d}t} \approx kP$.
- If $P > M$, then $1 - \frac{P}{M}$ is negative and so $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$.

Equation (2) is called *logistic differential equation* and we can solve the equation by the method of integration:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right) = \frac{k}{M}P\left(M - P\right) \Rightarrow$$

$\Rightarrow$

$\Rightarrow$

- If the initial population $P(0)$ lies between 0 and M, then $\frac{\mathrm{d}P}{\mathrm{d}t}$ _____, and the population ____________.
- If the population exceeds the carrying capacity ($P > M$), then $\frac{\mathrm{d}P}{\mathrm{d}t}$ _____, and the population ____________.
- In either case, if the population approached the carry capacity ($p \to M$), then $\frac{\mathrm{d}P}{\mathrm{d}t}$ _____, which means the population levels off (呈現平穩狀態).
- The graphs move away from the equilibrium solution $P = 0$ and move toward the equilibrium solution $P = M$.

Figure 1: Solutions of the logistic equation.

A Model for the Motion of a Spring, page 587

We consider the motion of an object with mass m at the end of a vertical spring. Hook's Law says that if the spring is stretched (or compressed) x units from its natural length, then it exerts a force that is proportional to x:

$$\text{restoring force} = -kx,$$

where k is a positive constant (called the *spring constant*, 彈性常數). If we ignore any external resisting force (due to air resistance or friction), by Newton's Second Law, we have

$$m\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx. \tag{3}$$

This is an example of a *second-order differential equation* (二階微分方程). All solutions of (3) can be written as

$$x(t) = A\sin kt + B\cos kt,$$

General Differential Equations, page 588

In general, a *differential equation* (微分方程) is an equation that contains an unknown function and one or more of its derivatives. The *order* of a differential equation is the order of the highest derivative that occurs in the equation.

A function $f(x)$ is called a *solution* of a differential equation (微分方程的解) if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation.

When apply differential equations, we are usually not as interested in finding a family of solution (the *general solution*, 一般解). In many problems we need to find the particular solution that satisfies a condition of the form $y(t_0) = y_0$. This is called an *initial condition* (初始條件), and the problem of finding a solution of the differential equation that satisfies the initial condition is called an *initial-value problem* (初始值問題).

Example 3 (page 591)**.** Psychologists interested in learning theory study *learning curves* (學習曲線). A learning curve is the graph of a function $P(t)$, the performance of someone learning a skill as a function of the training time t. The derivative $\frac{\mathrm{d}P}{\mathrm{d}t}$ represents the rate at which performance improves.

If M is the maximum level of performance of which the learner is capable, then

$$\frac{\mathrm{d}P}{\mathrm{d}t} = k(M - P(t))$$

is a model for learning, where k is a positive constant. Solve it as a linear differential equation and graph your Calculus learning curve.

Solution.