# Chapter 9 Differential Equations

## 9.1 Modeling with Differential Equations (page 586)

An ordinary differential equation (普通微分方程) is an equation that contains an unknown function and some of its derivatives. In a real world problem, we use the mathematical model in form of a differential equation because we often notice that changes occur and we want to predict future behavior on the basis of how current values change.

### Models of Population Growth, page 586

**Example 1** (page 586). One model for the growth of a population is based on the assumption that the population grows at a rate proportional to the size of the population. That is a reasonable assumption for a population of bacteria or animals under ideal condition (unlimited environment, adequate nutrition, absence of predators, immunity from disease.)

Let's identify and name the variables in this model:

- t = time.
- P = P(t) = the number of individuals in the population.

So we can get the differential equation:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP,\tag{1}$$

where k is the proportionality constant.

We know from Chapter 3 that exponential functions satisfy the equation (1). Here we can solve the equation by the method of integration:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP \Rightarrow$$

In this formula, we allow C to vary through all the real numbers, and we get the *family* of solution. But in real world problem, populations have only positive values and so we are interested only in the solutions with  $C_0 > 0$ . (We probably concerned only with values of t greater than the initial time t = 0.) We can put t = 0 and get  $P(0) = C_0 e^k \cdot 0 = C_0$ , so the constant  $C_0$  turns out to be the initial population P(0).

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**Example 2** (Logistic differential equation, page 587). Example 1 shows a model for population growth under ideal conditions, but we have to recognize that a more realistic model must reflect that the fact that a given environment has limited resources. Many populations starts by increasing in an exponential manner, but the population levels off when it approaches its carrying capacity M (or decreases toward M if it ever exceed M, 最大負荷量). For a model to take into account both trends we make to assumptions:

- $\frac{dP}{dt} \approx kP$  if P is small. (Initially, the growth rate is proportional to P).
- $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$  if P > M. (P decreases if it ever exceeds M.)

A simple expression that incorporates both assumptions is given by the equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right).\tag{2}$$

Why this differential equation is reasonable?

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- If P is small compared with M, then  $\frac{P}{M}$  is closed to 0 and so  $\frac{dP}{dt} \approx kP$ .
- If P > M, then  $1 \frac{P}{M}$  is negative and so  $\frac{dP}{dt} < 0$ .

Equation (2) is called *logistic differential equation* and we can solve the equation by the method of integration:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right) = \frac{k}{M}P\left(M - P\right) \Rightarrow$$
$$\Rightarrow$$

- If the initial population P(0) lies between 0 and M, then  $\frac{dP}{dt}$ , and the population \_\_\_\_\_
- If the population exceeds the carrying capacity (P > M), then  $\frac{dP}{dt}$ , and the population \_\_\_\_\_
- In either case, if the population approached the carry capacity  $(p \to M)$ , then  $\frac{dP}{dt}$ , which means the population levels off (呈現平穩狀態).
- The graphs move away from the equilibrium solution P = 0 and move toward the equilibrium solution P = M.

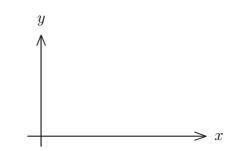


Figure 1: Solutions of the logistic equation.

### A Model for the Motion of a Spring, page 587

We consider the motion of an object with mass m at the end of a vertical spring. Hook's Law says that if the spring is stretched (or compressed) x units from its natural length, then it exerts a force that is proportional to x:

restoring force = -kx,

where k is a positive constant (called the *spring constant*, 彈性常數). If we ignore any external resisting force (due to air resistance or friction), by Newton's Second Law, we have

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx.\tag{3}$$

This is an example of a *second-order differential equation* (二階微分方程). All solutions of (3) can be written as

$$x(t) = A\sin kt + B\cos kt$$

#### General Differential Equations, page 588

In general, a *differential equation* (微分方程) is an equation that contains an unknown function and one or more of its derivatives. The *order* of a differential equation is the order of the highest derivative that occurs in the equation.

A function f(x) is called a *solution* of a differential equation (微分方程的解) if the equation is satisfied when y = f(x) and its derivatives are substituted into the equation.

When apply differential equations, we are usually not as interested in finding a family of solution (the general solution, 一般解). In many problems we need to find the particular solution that satisfies a condition of the form  $y(t_0) = y_0$ . This is called an *initial condition* (初始條件), and the problem of finding a solution of the differential equation that satisfies the initial condition is called an *initial-value* problem (初始值問題). **Example 3** (page 591). Psychologists interested in learning theory study *learning* curves (學習曲線). A learning curve is the graph of a function P(t), the performance of someone learning a skill as a function of the training time t. The derivative  $\frac{dP}{dt}$  represents the rate at which performance improves.

If M is the maximum level of performance of which the learner is capable, then

$$\frac{\mathrm{d}P}{\mathrm{d}t} = k(M - P(t))$$

is a model for learning, where k is a positive constant. Solve it as a linear differential equation and graph your Calculus learning curve.

#### Solution.