8.2 Area of a Surface of Revolution (page 551)

In this section, we will derive the formula of the area of a surface of revolution.

Example 1 (page 551)**.** Find the lateral surface area of a circular cylinder with radius r and height h.

Solution.

Example 2 (page 551)**.** Find the lateral surface area of a circular cone with base radius r and slant height l.

Solution.

Example 3 (page 551)**.** Find the lateral surface area of a *band*, which is a portion of a circular cone with upper radius r_1, lower radius r_2, and slant height l.

Solution.

Example 4 (page 552)**.** Find the surface area of the surface obtained by rotating the curve $y = f(x), a \le x \le b$, about the x-axis. (Assume that $f \in C^1[a, b]$.)

Solution.

(a) Partition: $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. We have $\Delta x = x_i - x_{i-1} = \frac{b-a}{n}$.

(b) Approximate the curve $(x, f(x))$ by polygons $\cup_{i=1}^n \overline{P_{i-1}P_i}$, where $P_i = (x_i, f(x_i))$.

(c) By the mean value theorem, the surface area is approximated by

$$\begin{aligned} S_n &= \sum_{i=1}^n 2\pi \left(\frac{f(x_{i-1}) + f(x_i)}{2} \right) |\overline{P_{i-1}P_i}| \\ &= \sum_{i=1}^n 2\pi \left(\frac{f(x_{i-1}) + f(x_i)}{2} \right) \sqrt{(f(x_i) - f(x_{i-1}))^2 + (x_i - x_{i-1})^2} \\ &= \sum_{i=1}^n 2\pi \left(f(x_i^*) + (f(x_i^{**}) - f(x_i^*)) \right) \sqrt{1 + (f'(x_i^*))^2}\, \Delta x, \end{aligned}$$

where $x_i^{**} \in [x_{i-1}, x_i]$ satisfies $f(x_i^{**}) = \frac{f(x_{i-1})+f(x_i)}{2}$ and $x_i^* \in [x_{i-1}, x_i]$.

(d) When $n \to \infty$, we have

$$\begin{aligned} \lim_{n\to\infty} S_n &= \lim_{n\to\infty} \sum_{i=1}^n 2\pi \left(f(x_i^*) + (f(x_i^{**}) - f(x_i^*)) \right) \sqrt{1 + (f'(x_i^*))^2}\, \Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^n 2\pi f(x_i^*) \sqrt{1 + (f'(x_i^*))^2}\, \Delta x \\ &\quad + \lim_{n\to\infty} \sum_{i=1}^n 2\pi (f(x_i^{**}) - f(x_i^*)) \sqrt{1 + (f'(x_i^*))^2}\, \Delta x \\ &= \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, \mathrm{d}x, \end{aligned}$$

where we need to estimate (use mean value theorem and extreme value theorem)

$$\begin{aligned} &\sum_{i=1}^n 2\pi \left| f(x_i^{**}) - f(x_i^*) \right| \sqrt{1 + (f'(x_i^*))^2}\, \Delta x \\ &\le \sum_{i=1}^n 2\pi |f'(x_i^{***}) \Delta x| \sqrt{1 + (f'(x_i^*))^2}\, \Delta x \\ &\le 2\pi M \sqrt{1 + M^2} \sum_{i=1}^n (\Delta x)^2 = 2\pi M \sqrt{1 + M^2} \sum_{i=1}^n \left(\frac{b-a}{n^2} \right)^2 \\ &= 2\pi M \sqrt{1 + M^2} (b-a)^2 \cdot \frac{1}{n} \to 0 \end{aligned}$$

as $n \to \infty$.

Definition 5. *Surface area* (表面積) of the surface obtained by rotating the curve $y = f(x), a \leq x \leq b$, about x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,\mathrm{d}x = \int_a^b 2\pi y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Recall that the differential of the arc length function is

$$(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 \Rightarrow \mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y. \tag{1}$$

If the curve is described as $x = g(y), c \leq y \leq d$, then the formula for surface area (rotating about x-axis) becomes

$$S = \int_c^d 2\pi y\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y. \tag{2}$$

Formula (1) and (2) can be formally written as

$$\int 2\pi y\,\mathrm{d}s.$$

For rotation about the y-axis, the surface area formula (formally) becomes

$$S = \int 2\pi x\,\mathrm{d}s, \quad \text{where} \quad \mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$$

☐ 表面積公式: 周長乘以弧長的積分。

Example 6. Find the area of surface obtained by rotating $y = \sin x, 0 \leq x \leq \pi$, about the x-axis.

Solution.

Example 7 (page 556)**.** Consider the region $\mathcal{R} = \{(x, y)|x \geq 1, 0 \leq y \leq \frac{1}{x}\}$ rotating about the x-axis.

(a) Show that the volume of the resulting solid is finite.

(b) Show that the surface area is infinite. (The surface is called *Gabriel's horn.*)

Solution.