Chapter 8 Further Applications of Integration

8.1 Arc Length (page 544)

The Arc Length Formula (page 544). *If f' is continuous on $[a,b]$, then the length of the curve $y = f(x), a \le x \le b$, is*

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x. \tag{1}$$

If a curve has the equation $x = g(y), c \le y \le d$, and $g'(y)$ is continuous, then by interchanging the roles of x and y, we obtain the following formula for its length:

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\,\mathrm{d}y = \int_c^d \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y. \tag{2}$$

Proof. Suppose that a curve C is defined by the equation $y = f(x)$, where $f(x)$ is continuous and $a \le x \le b$. We obtain a polygonal approximation to C by dividing the interval $[a,b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx. If $y_i = f(x_i)$, then the point $P_i(x_i, y_i)$ lies on C and the polygon with vertices $P_0, P_1, \ldots, P_n$ is an approximation to C.

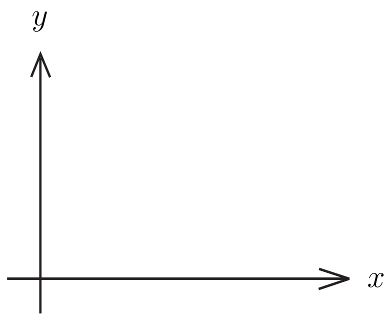

Figure 1: We use the length of inscribed polygons to approximate the length of C.

We define the *length* L (弧長) of the curve C with equation $y = f(x), a \le x \le b$, as the limit of the lengths of these inscribed polygons (if the limit exists):

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

When f' is continuous on $[a,b]$ (we say $f \in C^1([a,b])$), then by the Mean Value Theorem, there is a number $x_i^* \in (x_{i-1}, x_i)$ such that

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (f(x_i) - f(x_{i-1}))^2} \\ &= \sqrt{(\Delta x)^2 + (f'(x_i^*)\Delta x)^2} = \sqrt{1 + (f'(x_i^*))^2}\Delta x. \end{aligned}$$

Therefore,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| =$$

To get the formula (1), by similar discussion, we divide $c \le y \le d$ into n subinterval with endpoints $y_0, y_1, \ldots, y_n$ and equal width Δy, and rewrite $|P_{i-1}P_i|$ as

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(g(y_i) - g(y_{i-1}))^2 + (\Delta y)^2} \\ &= \sqrt{(g'(y_i^*)\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + (g'(y_i^*))^2}\Delta y, \end{aligned}$$

so

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| =$$

□

Example 1. Show that the circumference of a circle with radius r is $2\pi r$.

Solution.

Example 2. Find the length of the curve $x = \frac{1}{3}\sqrt{y}(y-3), 1 \le y \le 9$.

Solution.

Example 3 (page 549)**.** Find the length of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Solution.

Exercise. Let $f(x) = \ln x - \frac{1}{8}x^2, 1 \le x \le 2$. Find the length of the graph of f.

Exercise. Find the length of the curve $y = \int_1^{\frac{1}{x^2}} \sqrt{t^3+1}\,\mathrm{d}t$, $\frac{1}{2} \le x \le 1$.

Exercise. Given $0 < a < b$, find the arc length of $y = \ln\left(\frac{\mathrm{e}^x+1}{\mathrm{e}^x-1}\right)$ for $a \le x \le b$.

The Arc Length Function

We will find it useful to have a function that measures the arc length of a curve from a particular starting point to any other point on the curve. If a smooth curve C has the equation $y = f(x), a \le x \le b$, let $s(x)$ be the distance along C from the initial point $P_0(a, f(x))$ to the point $Q(x, f(x))$. Then

$$s(x) = \int_a^x \sqrt{1 + (f'(t))^2}\,\mathrm{d}t$$

is a function, called the *arc length function* (弧長函數).

By the Fundamental Theorem of Calculus, we get

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + (f'(x))^2} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$$

In differential sense, we can view the arc length as the infinitesimal Pythagorean theorem: $(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2$.

Similarly, we have

$$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y.$$

Remark 4. In general, we can viewed the curve as $\mathbf{r}(x) = (x, f(x))$. Then we have $\mathbf{r}'(x) = (1, f'(x))$ and $|\mathbf{r}'(x)| = \sqrt{1 + (f'(x))^2}$, so the arc-length formula becomes

$$s(x) = \int_a^x |\mathbf{r}'(t)|\,\mathrm{d}t.$$