7.8 Improper Integrals (page 527)

In this section we extend the concept of a definite integral to the case where the interval is infinite and also to the case where f has an infinite discontinuity in $[a, b]$. In either case the integral is called an *improper integral* (瑕積分).

Type 1: Infinite Intervals

Definition of an Improper Integral of Type 1 (page 528)**.**

(a) If $\int_a^t f(x)\,\mathrm{d}x$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty} \int_a^t f(x)\,\mathrm{d}x$$

provided this limits exists (as a finite number).

(b) If $\int_t^b f(x)\,\mathrm{d}x$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{t\to-\infty} \int_t^b f(x)\,\mathrm{d}x$$

provided this limits exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,\mathrm{d}x$ and $\int_{-\infty}^b f(x)\,\mathrm{d}x$ are called *convergent* (收斂) if the corresponding limits exists and *divergent* (發散) if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,\mathrm{d}x$ and $\int_{-\infty}^a f(x)\,\mathrm{d}x$ convergent, then we define

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^a f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x$$

In part (c) any real number a can be used.

Example 1 (page 530)**.** Discuss the areas of the infinite region $\mathcal{R}$ under the curve $y = \frac{1}{x^p}, p > 0$ and to the right $x = 1$.

Solution.

Type 2: Discontinuous Integrands

Definition of an Improper Integral of Type 2 (page 531)**.**

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to b^-} \int_a^t f(x)\,\mathrm{d}x$$

if this limits exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to a^+} \int_t^b f(x)\,\mathrm{d}x$$

if this limits exists (as a finite number).

The improper integrals $\int_a^b f(x)\,\mathrm{d}x$ is called *convergent* (收斂) if the corresponding limits exists and *divergent* (發散) if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,\mathrm{d}x$ and $\int_c^b f(x)\,\mathrm{d}x$ convergent, then we define

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

Example 2 (page 535)**.** Discuss the areas of the region $\mathcal{R}$ under the curve $y = \frac{1}{x^p}, p > 0$, and between $x = 0$ and $x = 1$.

Solution.

Example 3. Compare Example 1 with Example 2.

Solution.

A Comparison Test for Improper Integrals

Comparison Theorem (page 533)**.** *Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.*

(a) *If $\int_a^\infty f(x)\,\mathrm{d}x$ is convergent, then $\int_a^\infty g(x)\,\mathrm{d}x$ is convergent.*

(b) *If $\int_a^\infty g(x)\,\mathrm{d}x$ is divergent, then $\int_a^\infty f(x)\,\mathrm{d}x$ is divergent.*

☐ 定理的條件「$f(x) \geq g(x) \geq 0$」, 函數「非負」是必要的。

☐ 定理敘述中「for $x \geq a$」可以改成「for some $x \geq b$, $b \geq a$」。

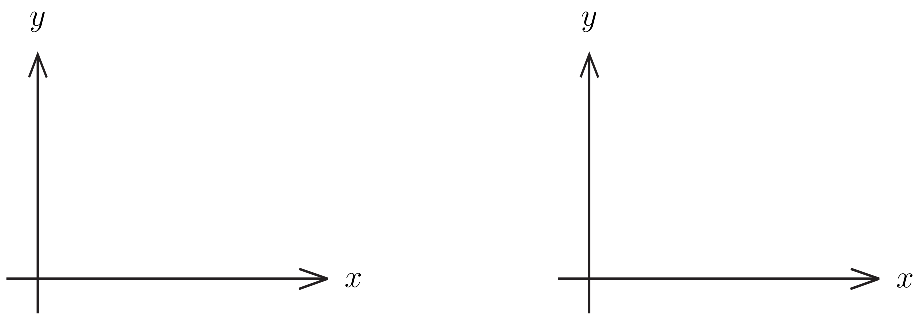

Figure 1: Comparison Theorem.

Example 4.

(a) Find the values of α for which the improper integral $\displaystyle\int_1^\infty \frac{1}{x^\alpha(1+\sqrt{x})}\,\mathrm{d}x$ converges.

(b) Evaluate the integral $\displaystyle\int_0^\pi \sec^2 x\,\mathrm{d}x$.

Solution.

Example 5 (page 535)**.** Let $I_n = \displaystyle\int_0^{\infty} x^n \mathrm{e}^{-x}\,\mathrm{d}x$. Find the reduction formula.

Solution.

Example 6 (page 535)**.**

(a) Determine the values of $\alpha > 0$ such that $\displaystyle\int_1^{\infty} \frac{\ln x}{x^{\alpha}}\,\mathrm{d}x$ is convergent.

(b) Find the integral $\displaystyle\int_1^{\infty} \frac{\ln x}{x^3}\,\mathrm{d}x$.

Solution.

Example 7. Evaluate the improper integral $\displaystyle\int_0^{2} \frac{\sqrt{x(2-x)}}{x}\,\mathrm{d}x$.

Solution.