7.5 Strategy for Integration (page 503)

We have learned the following techniques to integrate a function:

- Substitution rule, section 5.5.
- Integration parts, section 7.1.
- Trigonometric Integrals 7.2.
- Trigonometric substitution, section 7.3.
- Partial fractions, section 7.4.

In this section, we present a collection of miscellaneous integrals in random and the main challenge is to recognize which technique or formula to use.

Table of Indefinite Integrals (page 503)**.**

$$\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}\left(\frac{x}{a}\right) + C, \quad a > 0$$

$$\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C \qquad \int \frac{1}{\sqrt{x^2 \pm a^2}}\,\mathrm{d}x = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$$

Once you are armed with these basic integration formulae, if you don't immediately see how to attack a given integral, you might try the following four-steps strategy.

(1) Simplify the integrand if possible. Use algebraic manipulation or trigonometric identities to simplify the integrand.

$$\int \sqrt{x}(1+\sqrt{x})\,\mathrm{d}x =$$

$$\int \frac{\tan\theta}{\sec^2\theta}\,\mathrm{d}\theta =$$

$$\int (\sin\theta + \cos\theta)^2\,\mathrm{d}\theta =$$

(2) Look for an obvious substitution.

$$\int \frac{x}{x^2-1}\,\mathrm{d}x =$$

(3) Classify the integrand according to its form.

(a) Trigonometric function: product of powers of $\sin x$ and $\cos x$, of $\tan x$ and $\sec x$, or $\cot x$ and $\csc x$.

(b) Rational functions: $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

(c) Integration by parts: product of a power of a polynomial and a transcendental function (trigonometric, exponential, or logarithmic).

(d) Radicals: $\sqrt{\pm x^2 \pm a^2}$, $\sqrt[n]{ax+b}$, or $\sqrt[n]{g(x)}$.

(4) Try again: remember that there are basically only two methods of integration: substitution and parts.

(a) Try substitution: inspiration, ingenuity, desperation.

(b) Try parts: it is sometimes effective on single function, such as $\sin^{-1} x$, $\tan^{-1} x$, $\ln x$ (inverse functions).

(c) Manipulate the integrand:

$$\int \frac{1}{1-\cos x}\,\mathrm{d}x =$$

$$=$$

(d) Relate the problem to previous problems.

$$\int \tan^2 x \sec x \,\mathrm{d}x =$$

(e) Use several methods: substitution, integration by parts, etc.

Example 1 (page 505)**.** Compute $\displaystyle\int \frac{\tan^3 x}{\cos^3 x}\,\mathrm{d}x$.

Solution.

Solution 2.

Example 2 (page 506)**.** Compute $\displaystyle\int \mathrm{e}^{\sqrt{x}}\,\mathrm{d}x$.

Solution.

Example 3 (page 506)**.** Compute $\displaystyle\int \sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x$.

Solution.

Can we integrate all continuous functions?

Definition 4 (page 506)**.** *Elementary functions* are all polynomials, rational functions, power functions, exponential functions, logarithmic functions, trigonometric and inverse trigonometric functions, hyperbolic and inverse hyperbolic functions, and all functions that can be obtained from these by the five operations of addition, substraction, multiplication, division, and composition.

If $f(x)$ is an elementary function, then $f'(x)$ is an elementary function, but $\int f(x)\,\mathrm{d}x$ need not be an elementary function. For example,

(1) $\displaystyle\int \sqrt{1-2\sin^2 x}\,\mathrm{d}x$: elliptic integral (橢圓積分), 它是計算橢圓弧長的積分表達式 (8.1 會介紹如何計算曲線的長度)。

(2) $\displaystyle\frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t$: error function (誤差函數), 常用於機率統計 (常態分布) 與工程。

(3) $\displaystyle\int \sin(x^2)\,\mathrm{d}x, \int \cos(x^2)\,\mathrm{d}x$: Fresnel integral (菲涅耳積分), 與誤差函數有關聯。

(4) $\displaystyle\int \frac{\sin x}{x}\,\mathrm{d}x, \int \cos(\mathrm{e}^x)\,\mathrm{d}x$: sine integral function (cosine integral function)。

(5) $\displaystyle\int \frac{\mathrm{e}^x}{x}\,\mathrm{d}x$: exponential integral (指數積分)。

(6) $\displaystyle\int \frac{1}{\ln x}\,\mathrm{d}x$: logarithmic integral (對數積分)。

(7) $\displaystyle\int \sqrt{x^3+1}\,\mathrm{d}x$: 此積分可以化簡成和橢圓積分有關。

雖然上述函數的積分無法表示成基本函數的型式, 但是有時代入特殊的上、下限可以透過其他分析方式 (多變數微積分、複變函數論、微分方程) 等求得明確的數值。例如:

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}, \quad \int_{-\infty}^{\infty} \sin(x^2)\,\mathrm{d}x = \sqrt{\frac{\pi}{2}}, \quad \int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \pi.$$