7.4 Integration of Rational Functions by Partial Fractions (page 493)

In this section we show how to integrate any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called *partial fractions* (部份分式).

Example 1. Discuss the integral $\displaystyle\int \frac{1}{(ax+b)^k}\,\mathrm{d}x$, where $k \in \mathbb{N}$.

Solution.

Example 2. Discuss the integral $\displaystyle\int \frac{Ax+B}{(ax^2+bx+c)^k}\,\mathrm{d}x$, where $b^2-4ac<0, k\in\mathbb{N}$.

Solution.

Integrate rational functions

Step 1: Perform the long division (長除法)

Definition 3 (page 494)**.** Consider a rational function $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

(a) If the degree of $P(x)$ is less than the degree of $Q(x)$, such a rational function $f(x)$ is called *proper.*

(b) If the degree of $P(x)$ is greater or equal to the degree of $Q(x)$, such a rational function $f(x)$ is called *improper.*

If $f(x)$ is *improper*, then we use the long division to get

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

and $\frac{R(x)}{Q(x)}$ is proper.

Step 2: Factor the denominator $Q(x)$ as a product of linear factors $(ax+b)$ and irreducible quadratic factors $ax^2 + bx + c,\ b^2 - 4ac < 0$. (因式分解)

Step 3: Express the proper rational function $\frac{R(x)}{Q(x)}$ as a sum of partial fractions. (拆成部份分式, 不同類型有不同的拆解法)

Definition 4 (page 494)**.** A rational function is called a *partial fraction* if it is of the form

$$\frac{A}{(ax+b)^n} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^n}.$$

(1) *$Q(x)$ is a product of distinct linear factors.* That is,

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k),$$

where *no* factor is repeated, then there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}.$$

For example, $\dfrac{2x^2 - 3x - 1}{x(x+1)(x-1)} =$ ____________________, then

(2) *$Q(x)$ is a product of linear factors, some of which are repeated.* Suppose the first linear factor (a_1x+b_1) is repeated r times, then instead of the single term $\frac{A_1}{a_1x+b_1}$, we would use

$$\frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_1x+b_1)^2}+\cdots+\frac{A_r}{(a_1x+b_1)^r}.$$

For example, $\dfrac{2x^2-x+3}{(x-1)^3} =$ ____ .

(3) *$Q(x)$ contains irreducible quadratic factors, none of which is repeated.* That is, $Q(x)$ has the factor ax^2+bx+c, where $b^2-4ac<0$, then the expression for $\frac{R(x)}{Q(x)}$ will have a term of the form

$$\frac{Ax+B}{ax^2+bx+c},$$

and then we will use the formula

$$\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C.$$

(4) *$Q(x)$ contains a repeated irreducible quadratic factor.* If $Q(x)$ has the factor $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then instead of the single partial fraction, the sum

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

occurs in the $\frac{R(x)}{Q(x)}$.

For example, $\dfrac{x^3-x^2+2x+2}{(x^2+1)^2} =$ ____ .

Example 5 (`Case` (1))**.** Show that $\int \sec\theta\,\mathrm{d}\theta = \ln|\sec\theta+\tan\theta|+C$.

Solution.

Example 6 (`Case` (2))**.** Find the integral $\displaystyle\int \frac{x^2+3x+2}{x^3-3x+2}\,\mathrm{d}x$.

Solution.

Exercise. Evaluate $\displaystyle\int_1^2 \frac{x^2+4}{x^4+3x^3+2x^2}\,\mathrm{d}x$.

Example 7 (`Case` (3))**.** Evaluate the integral $\displaystyle\int \frac{2x^2+5x+3}{(x^2+2x+2)(x-1)}\,\mathrm{d}x$.

Solution.

Exercise. Evaluate the integral $\displaystyle\int \frac{1}{x^3+1}\,\mathrm{d}x$.

Exercise. Find the integral $\displaystyle\int \frac{1}{x^3+x^2+x}\,\mathrm{d}x$.

Exercise. Evaluate the integral $\displaystyle\int \frac{1}{x^3-1}\,\mathrm{d}x$.

Exercise. Evaluate $\displaystyle\int \frac{x^3-x^2+x+1}{x^3+x^2+x+1}\,\mathrm{d}x$.

Example 8. Find the integral $\displaystyle\int \frac{2x+1}{(x^2+1)^2}\,\mathrm{d}x$.

Solution.

Rationalizing substitutions, page 500

Some nonrational functions can be changed into rational functions by means of appropriate substitutions. In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, then the substitution $u = \sqrt[n]{g(x)}$ may be effective.

Example 9 (page 500)**.** Evaluate $\displaystyle\int \frac{\sqrt{x+4}}{x}\,\mathrm{d}x$.

Solution.

Exercise. Evaluate the integral $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{x}\,\mathrm{d}x$.

Convert rational functions of $\sin x$ and $\cos x$, page 502

The German mathematician Karl Weierstrass (1815-1897) noticed that the substitution $t = \tan(\frac{x}{2})$ will convert any rational function of $\sin x$ and $\cos x$ into an ordinary rational function of t.

If $t = \tan(\frac{x}{2})$, $-\pi < x < \pi$, then we have

$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}} \quad \text{and} \quad \sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}.$$

By double-angle formula, we get

$$\cos x = \frac{1-t^2}{1+t^2} \quad \text{and} \quad \sin x = \frac{2t}{1+t^2}.$$

Furthermore, we can compute

$$\mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t.$$

☐ 被積函數是 $\sin x$ 與 $\cos x$ 組成的有理函數, 可透過變數代換 $t = \tan(\frac{x}{2})$ 處理。

Example 10. Find the integral $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{2+\cos x}\,\mathrm{d}x$.

Solution.

Exercise. Find the integral $\displaystyle\int \frac{1}{2+\sin x}\,\mathrm{d}x$.

Exercise. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{3-5\sin x}\,\mathrm{d}x$.