7.2 Trigonometric Integrals (page 479)

In this section we use trigonometric identities to integrate certain combinations of trigonometric functions.

Example 1 (page 481)**.** Evaluate $\displaystyle\int \sin^m x \cos^n x \,\mathrm{d}x$, where $m, n \geq 0$ are integers.

Solution.

(a) If $\underline{m = 2k+1}$, then

$$\int \sin^m x \cos^n x \,\mathrm{d}x = \int \sin^{2k+1} x \cos^n x \,\mathrm{d}x = -\int \sin^{2k} x \cos^n x \,\mathrm{d}\cos x$$

$$= -\int (1-\cos^2 x)^k \cos^n x \,\mathrm{d}\cos x = -\int \sum_{i=0}^{k} C_i^k 1^{k-i}(-1)^i \cos^{2i} x \cos^n x \,\mathrm{d}\cos x$$

$$= \sum_{i=0}^{k} (-1)^{i+1} C_i^k \int \cos^{n+2i} x \,\mathrm{d}(\cos x) = \sum_{i=0}^{k} \frac{(-1)^{i+1} C_i^k}{n+2i+1} \cos^{n+2i+1} x + C.$$

(b) If $\underline{n = 2k+1}$, then

$$\int \sin^m x \cos^n x \,\mathrm{d}x = \int \sin^m x \cos^{2k+1} x \,\mathrm{d}x = \int \sin^m x \cos^{2k} x \,\mathrm{d}(\sin x)$$

$$= \int \sin^m x (1-\sin^2 x)^k \,\mathrm{d}\sin x = \int \sin^m x \sum_{i=0}^{k} C_i^k 1^{k-i}(-1)^i \sin^{2i} x \,\mathrm{d}\sin x$$

$$= \sum_{i=0}^{k} (-1)^i C_i^k \int \sin^{m+2i} x \,\mathrm{d}\sin x = \sum_{i=0}^{k} \frac{(-1)^i C_i^k}{m+2i+1} \int \sin^{m+2i+1} x + C.$$

(c) If $\underline{m = 2k, n = 2l}$, then using the half-angle identities

$$\sin^2 x = \frac{1-\cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1+\cos 2x}{2},$$

we have

$$\int \sin^m x \cos^n x \,\mathrm{d}x = \int \sin^{2k} x \cos^{2l} x \,\mathrm{d}x$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^k \left(\frac{1+\cos 2x}{2}\right)^l \mathrm{d}x = \sum_{i=0}^{k} \sum_{j=0}^{l} \frac{(-1)^i C_i^k C_j^l}{2^{k+l}} \int \cos^{i+j} 2x \,\mathrm{d}x.$$

If $\underline{i+j \text{ is odd}}$, we reduce the integral to case (b).
If $\underline{i+j \text{ is even}}$, we use half-angle identities again.

Exercise (page 479)**.** Evaluate $\displaystyle\int \sin^5 x \cos^2 x \,\mathrm{d}x$.

Example 2 (page 480)**.** Evaluate $\displaystyle\int \sin^2 x \, \mathrm{d}x$.

Solution.

Example (TA) 3 (page 475, 477, 478)**.** Show that the following reduction formulae:

(a) $\displaystyle\int \sin^n x \, \mathrm{d}x = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x \, \mathrm{d}x, \quad n \geq 2.$

(b) $\displaystyle\int \cos^n x \, \mathrm{d}x = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x \, \mathrm{d}x, \quad n \geq 2.$

Solution.

Example (TA) 4. Find $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, \mathrm{d}x$ and $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{2n} x \, \mathrm{d}x$.

Solution.

Exercise (page 480)**.** Evaluate $\displaystyle\int \sin^4 x \, \mathrm{d}x$.

Example 5 (page 482)**.** Compute the integrals $\int \tan x \, \mathrm{d}x$ and $\int \sec x \, \mathrm{d}x$.

Solution.

□ 上述方法太過技巧 (誰知道要乘什麼量), 但之後會學別的方式處理它 (想法比較自然)。

Example (TA) 6 (page 477)**.** Show that the following reduction formulae:

(a) $\displaystyle\int \tan^n x \, \mathrm{d}x = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} \, \mathrm{d}x, \quad n \neq 1.$

(b) $\displaystyle\int \sec^n x \, \mathrm{d}x = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, \mathrm{d}x, \quad n \neq 1.$

Solution.

Exercise (page 483)**.** Evaluate $\int \tan^3 x \, \mathrm{d}x$.

Exercise (page 483)**.** Evaluate $\int \sec^3 x \, \mathrm{d}x$.

Example 7 (page 482)**.** Evaluate $\int \tan^m x \sec^n x \,\mathrm{d}x$, where $m, n \in \mathbb{N}$.

Solution.

(a) If $\underline{n = 2k, k \in \mathbb{N}}$, then

$$\int \tan^m x \sec^n x \,\mathrm{d}x = \int \tan^m x \sec^{2k} x \,\mathrm{d}x = \int \tan^m x \sec^{2k-2} x \,\mathrm{d}\tan x$$

$$= \int \tan^m x (\tan^2 x + 1)^{k-1} \,\mathrm{d}\tan x = \int \tan^m x \sum_{i=0}^{k-1} C_i^{k-1} \tan^{2i} x \,\mathrm{d}\tan x$$

$$= \sum_{i=0}^{k-1} C_i^{k-1} \int \tan^{m+2i} x \,\mathrm{d}\tan x = \sum_{i=0}^{k-1} \frac{C_i^{k-1}}{m+2i+1} \tan^{m+2i+1} x + C.$$

(b) If $\underline{m = 2k+1, k \in \mathbb{N}}$, then

$$\int \tan^m x \sec^n x \,\mathrm{d}x = \int \tan^{2k+1} x \sec^n x \,\mathrm{d}x = \int \tan^{2k} x \sec^{n-1} x \,\mathrm{d}\sec x$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \,\mathrm{d}\sec x = \int \sum_{i=0}^{k} C_i^k \sec^{2(k-i)} (-1)^i \sec^{n-1} x \,\mathrm{d}\sec x$$

$$= \sum_{i=0}^{k} (-1)^i C_i^k \int \sec^{2(k-i)+n-1} x \,\mathrm{d}\sec x = \sum_{i=0}^{k} \frac{(-1)^i C_i^k}{2(k-i)+n} \sec^{2(k-i)+n} x + C.$$

(c) If $\underline{m = 2k, n = 2l+1}$, then

$$I_m = \int \tan^m x \sec^n x \,\mathrm{d}x = \int \tan^{2k} x \sec^{2l+1} x \,\mathrm{d}x = \int \tan^{2k-1} x \sec^{2l} x \,\mathrm{d}\sec x$$

$$= \tan^{2k-1} x \sec^{2l+1} x - \int \sec x \,\mathrm{d}(\tan^{2k-1} x \sec^{2l} x)$$

$$= \tan^{2k-1} x \sec^{2l+1} x - \int \sec x (2k-1) \tan^{2k-2} \sec^2 x \sec^{2l} x \,\mathrm{d}x$$

$$- \int \sec x \tan^{2k-1} x (2l) \sec^{2l-1} x \sec x \tan x \,\mathrm{d}x$$

$$= \tan^{2k-1} x \sec^{2l+1} x - (2k-1) \int \tan^{2k-2} \sec^{2l+3} x \,\mathrm{d}x$$

$$- 2l \int \tan^{2k} x \sec^{2l+1} x \,\mathrm{d}x$$

$$= \tan^{2k-1} x \sec^{2l+1} x - (2k-1) \int \tan^{2k-2} (\tan^2 x + 1) \sec^{2l+1} x \,\mathrm{d}x$$

$$- 2l \int \tan^{2k} x \sec^{2l+1} x \,\mathrm{d}x$$

$$= \tan^{m-1} x \sec^n x - (m-1) I_m - (m-1) I_{m-2} - (n-1) I_m.$$

Hence we get

$$I_m = \frac{1}{m+n-1}\left(\tan^{m-1} x \sec^n x - (m-1)I_{m-2}\right),$$

and the reduction formula will reduce the integral to $\int \sec^n x \,\mathrm{d}x = \int \sec^{2l+1} x \,\mathrm{d}x$, $l \in \mathbb{N}$.

Exercise (page 481)**.** Evaluate $\displaystyle\int \tan^6 x \sec^4 x \,\mathrm{d}x$.

Exercise (page 482)**.** Evaluate $\displaystyle\int \tan^5 x \sec^7 x \,\mathrm{d}x$.

Example 8 (page 484)**.** Evaluate the following integrals:

$$\int \sin mx \cos nx \,\mathrm{d}x; \quad \int \cos mx \cos nx \,\mathrm{d}x; \quad \int \sin mx \sin nx \,\mathrm{d}x.$$

Solution. Recall the following identities:

$$\begin{aligned} \sin x \cos y &= \frac{1}{2}(\sin(x+y) + \sin(x-y)) \\ \cos x \cos y &= \frac{1}{2}(\cos(x+y) + \cos(x-y)) \\ \sin x \sin y &= -\frac{1}{2}(\cos(x+y) - \cos(x-y)). \end{aligned}$$

If $\underline{m+n \neq 0}$ and $\underline{m-n \neq 0}$, then

$$\int \sin mx \cos nx \,\mathrm{d}x = \frac{1}{2}\int \sin((m+n)x) + \sin((m-n)x) \,\mathrm{d}x$$

$$= \begin{cases} \\ \\ \end{cases}$$

$$\int \cos mx \cos nx \,\mathrm{d}x = \frac{1}{2}\int \cos((m+n)x) + \cos((m-n)x) \,\mathrm{d}x$$

$$= \begin{cases} \\ \\ \end{cases}$$

$$\int \sin mx \sin nx \,\mathrm{d}x = -\frac{1}{2}\int \cos((m+n)x) - \cos((m-n)x) \,\mathrm{d}x$$

$$= \begin{cases} \\ \\ \end{cases}$$

In particular,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin mx \cos nx \,\mathrm{d}x = \left\{ \begin{array}{l} \\ \\ \end{array} \right.$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos mx \cos nx \,\mathrm{d}x = \left\{ \begin{array}{l} \\ \\ \end{array} \right.$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin mx \sin nx \,\mathrm{d}x = \left\{ \begin{array}{l} \\ \\ \end{array} \right.$$

Hence $\{\sin mx, m \in \mathbb{N}$ and $\cos nx, n \in \mathbb{Z}, n \geq 0\}$ form an "orthonormal basis" in the function space $C[-\pi, \pi]$.