# Chapter 7 Techniques of Integration

## 7.1 Integration by Parts (page 472)

The rule that corresponds to the Product Rule for differentiation is called the rule for *integration by parts* (分佈積分).

The Product Rule states that if $f$ and $g$ are differentiable functions, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x).$$

In the notation for indefinite integrals this equation becomes

$$\int (f(x)g'(x) + g(x)f'(x))\,\mathrm{d}x = f(x)g(x), \quad \text{or}$$
$$\int f(x)g'(x)\,\mathrm{d}x + \int g(x)f'(x)\,\mathrm{d}x = f(x)g(x).$$

So we can rearrange this equation as

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x. \tag{1}$$

Formula (1) is called the *formula for integration by parts*.

Let $u = f(x)$ and $v = g(x)$, then the differentials are $\mathrm{d}u = f'(x)\,\mathrm{d}x$ and $\mathrm{d}v = g'(x)\,\mathrm{d}x$. By the Substitution Rule, the formula for integration by parts becomes

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u. \tag{2}$$

**Example 1** (page 472)**.** Evaluate $\int x\sin x\,\mathrm{d}x$.

**Solution.**

**Exercise.** Find $\int x^2\sin(3x)\,\mathrm{d}x$.

☐ 遞迴公式: 對所有 $n \in \mathbb{N}$, $\int x^n \sin x\,\mathrm{d}x = -x^n\cos x + n\int x^{n-1}\cos x\,\mathrm{d}x$。

☐ 分佈積分後可以讓多項式的次數降一次; 降到零次 (常數) 便可以積分。

**Example 2** (page 473)**.** Evaluate $\int \ln x \, \mathrm{d}x$.

**Solution.**

**Exercise.** Evaluate $\int (\ln x)^2 \, \mathrm{d}x$.

☐ 遞迴公式: 對所有 $n \in \mathbb{N}$, $\int (\ln x)^n \, \mathrm{d}x = x(\ln x)^n - n \int (\ln x)^{n-1} \, \mathrm{d}x$.

☐ 分佈積分後要讓 $\ln x$ 微分變成 $\frac{1}{x}$ 之後的積分式便能處理。

**Example 3.** Find the integral $\int_1^{\mathrm{e}} x(\ln x)^2 \, \mathrm{d}x$.

**Solution.**

**Exercise.** Evaluate the integral $\int x^\alpha \ln x \, \mathrm{d}x, \alpha \in \mathbb{R}$.

**Example 4.** Evaluate $\int x \, \mathrm{e}^x \, \mathrm{d}x$.

**Solution.**

**Exercise** (page 474)**.** Evaluate $\int x^2 \, \mathrm{e}^x \, \mathrm{d}x$.

☐ 遞迴公式: 對所有 $n \in \mathbb{N}$, $\int x^n \, \mathrm{e}^x \, \mathrm{d}x = x^n \mathrm{e}^x - n \int x^{n-1} \, \mathrm{e}^x \, \mathrm{d}x$.

☐ 分佈積分後可以讓多項式的次數降一次; 降到零次 (常數) 便可以積分。

**Example 5** (page 474)**.** Evaluate $\int \mathrm{e}^x \sin x \, \mathrm{d}x$.

**Solution.**

☐ $\sin x$ 和 $\cos x$ 的微分有週期性, 利用兩次分佈積分之後再將整個等式處理。

**Example 6** (page 475)**.** Evaluate $\int \tan^{-1} x \, \mathrm{d}x$.

**Solution.**

☐ 反三角函數的積分, 直接用分佈積分法。

**Exercise.** Evaluate the integral $\int \mathrm{e}^{\sin^{-1} x} \, \mathrm{d}x$.

**Exercise.** Evaluate $\int_0^1 x \tan^{-1}(x^2) \, \mathrm{d}x$ and $\int_0^1 x(\tan^{-1} x)^2 \, \mathrm{d}x$

**Example 7.** Consider the region $\mathcal{R}$ enclosed by the curves $y = \cos x$ and $y = \sin x$, and $0 \le x \le \frac{\pi}{4}$. Find the volume of the solid obtained by rotating the region about the $y$-axis.

**Solution.**

**Exercise.** Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

(a) $y = 1 + \sin x$, $y = 0$, and $0 \leq x \leq 2\pi$; about the $y$-axis.

(b) $y = \cos x, y = 0$, and $0 \leq x \leq \frac{\pi}{2}$; about the $y$-axis.

**Example 8.** Find the average of the horizontal chords in $y = \sin x, 0 \leq x \leq \pi$.

**Solution.**