5.5 The Substitution Rule (page 413)

The Substitution Rule (page 413)**.** *If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then*

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(g(x))\,\mathrm{d}g(x) = \int f(u)\,\mathrm{d}u.$$

Proof. Suppose F is an antiderivative of f, then we have

$$\frac{\mathrm{d}}{\mathrm{d}x}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x).$$

So $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$. Let $u = g(x)$, then

$$\int f(g(x))g'(x)\,\mathrm{d}x = F(g(x)) + C = F(u) + C = \int F'(u)\,\mathrm{d}u = \int f(u)\,\mathrm{d}u.$$

The middle formula comes from the definition of differential: $\mathrm{d}g(x) = g'(x)\,\mathrm{d}x$. □

The Substitution Rule for Definite Integrals (page 416)**.** *If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then*

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u.$$

Proof. Let F be an antiderivative of f. Then $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$, by Part 2 of the Fundamental Theorem, we have

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = F(g(x))\Big|_a^b = F(g(b)) - F(g(a)).$$

On the other hand, for the right hand side of the equation, we have

$$\int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u = F(u)\Big|_{g(a)}^{g(b)} = F(g(b)) - F((a)).$$

□

Example 1. Compute the integral $\displaystyle\int \frac{x^7}{\sqrt{x^4+1}}\,\mathrm{d}x$.

Solution.

Example 2 (page 410)**.** Calculate $\int \tan x \, \mathrm{d}x$.

Solution.

Example 3. Find $\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sqrt{1+\tan\theta}} \mathrm{d}\theta$.

Solution.

Example 4. Find $\int \cos(ax+b)\, \mathrm{d}x$.

Solution.

Example 5. Find $\int \frac{(\ln x)^k}{x}\, \mathrm{d}x$.

Solution.

Example 6. Find the integral $\int \frac{\mathrm{e}^{\sin x}}{\sec x}\, \mathrm{d}x$.

Solution.

Example 7. Let $F(x) = \int_0^x (x-t)t\sin(t^2)\,\mathrm{d}t$. Find $F'(x)$.

Solution.

Example 8. Compute the integral $\displaystyle\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{\sin^{-1}\sqrt{x}}{\sqrt{x(1-x)}}\,\mathrm{d}x$.

Solution.

Integrals of Symmetric Functions (page 417)**.** *Suppose f is continuous on $[-a,a]$.*

(a) *If f is even, then* $\displaystyle\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x$.

(b) *If f is odd, then* $\displaystyle\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$.

Proof. We compute

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{-a}^{0} f(x)\,\mathrm{d}x + \int_0^a f(x)\,\mathrm{d}x = -\int_0^{-a} f(x)\,\mathrm{d}x + \int_0^a f(x)\,\mathrm{d}x,$$

Let $u=-x$, then $\mathrm{d}u = -\mathrm{d}x$ and when $x=-a$, $u=a$. Therefore

$$-\int_0^{-a} f(x)\,\mathrm{d}x = -\int_0^a f(-u)(-\mathrm{d}u) = \int_0^a f(-u)\,\mathrm{d}u.$$

(a) If f is even, then ____________, so we get

$$\int_{-a}^{a} f(x)\,\mathrm{d}x =$$

$$=$$

(b) If f is odd, then ______________, so we get

$$\int_{-a}^{a} f(x)\,\mathrm{d}x =$$

□

Appendix

Suppose that $f(x) \in C^1([a,b])$, which implies $|f'(x)| \leq M$. Let $\Delta x = \frac{b-a}{n}$, then

$$\left| \int_a^b f(x)\,\mathrm{d}x - \sum_{i=1}^n f(x_i^*)\Delta x \right| \leq \sum_{i=1}^n \left| \max_{[x_{i-1},x_i]} f(x) - \min_{[x_{i-1},x_i]} f(x) \right| \Delta x$$

$$\leq \sum_{i=1}^n |f'(\xi_i)|(\Delta x)^2 = M \cdot \sum_{i=1}^n \frac{(b-a)^2}{n^2} = M \cdot \frac{(b-a)^2}{n} \to 0 \text{ as } n \to \infty.$$

So for integration, before we take summation, the $\frac{1}{n}$ part is the whole material. We can ignore higher order term such as $\frac{1}{n^2}$ because it tends to zero after summation and n tends to infinity.

Therefore, for the Substitution Rule, we only focus on the "differentials" between two variables y and x. That is, $\mathrm{d}y = y'(x)\,\mathrm{d}x$ will catch all information of $\frac{1}{n}$ part between variables y and x.