5.3 The Fundamental Theorem of Calculus (page 392)

The Fundamental Theorem of Calculus, Part 1 (page 394)**.** *If f is continuous on $[a, b]$, then the function g defined by*

$$g(x) = \int_a^x f(t)\,\mathrm{d}t \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

Proof. For x and $x + h$ in (a, b), we have

$$g(x + h) - g(x) =$$

$$=$$

so for $h \neq 0$,

$$\frac{g(x + h) - g(x)}{h} =$$

Assume that $h > 0$. Since f is continuous on $[x, x+h]$, the ____________________ says that there are $u, v \in [x, x + h]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x, x + h]$. So

$$mh \leq \int_x^{x+h} f(t)\,\mathrm{d}t \leq Mh \Rightarrow$$

Now we let $h \to 0$, then $u \to x$ and $v \to x$, so

$$\lim_{h\to 0} f(u) = f(\lim_{u\to x} u) = f(x) \quad \text{and} \quad \lim_{h\to 0} f(v) = f(\lim_{v\to x} v) = f(x)$$

because f is continuous at x. By the ______________, we have

$$g'(x) = \lim_{h\to 0} \frac{g(x + h) - g(x)}{h} = f(x).$$

If $x = a$ or b, the above discussion can be modified by considering one-sided limit. Since g is differentiable on $[a, b]$, g is continuous on $[a, b]$. □

Remark 1. The Fundamental Theorem of Calculus, Part 1, can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x)$$

when f is continuous.

Example 2 (page 395)**.** Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\,\mathrm{d}t$.

Solution.

Example 3. Find the derivative of $h(x) = \int_0^{\sin x} \sqrt{1+r^3}\,\mathrm{d}r$.

Solution.

The Fundamental Theorem of Calculus, Part 2 (page 396)**.** *If f is continuous on $[a,b]$, then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a),$$

where $F(x)$ is any antiderivative of $f(x)$, that is, a function such that $F'(x) = f(x)$.

Proof. Let $g(x) = \int_a^x f(t)\,\mathrm{d}t$. From the Fundamental Theorem of Calculus, Part 1, we know $g'(x) = f(x)$, so $g(x)$ is an antiderivative of f. If F is any other antiderivative of f on $[a,b]$, then F and g differ by a constant: $F(x) = g(x) + C$ for $a < x < b$. Remark that it also holds when $x = a$ and $x = b$.

We put $x = a$ in the formula of $g(x)$ to get

So we have

$F(b) - F(a) =$ ____ .

□

□ We often use notation $F(x)|_a^b = F(b) - F(a)$.

The Fundamental Theorem of Calculus (page 398)**.** *Suppose f is continuous on $[a,b]$.*

(1) *If $g(x) = \int_a^x f(t)\,\mathrm{d}t$, then $g'(x) = f(x)$.*

(2) $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$, *where F is any antiderivative of f, that is, $F' = f$.*