5.2 The Definite Integral (page 378)

Definition of a Definite Integral (page 378)**.** If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n-subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0 = a, x_1, x_2, \dots, x_n = b$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \dots, x_n^*$ be any *sample points* (樣本點) in these subintervals, so x_i^* lies in the i-th subinterval $[x_{i-1}, x_i]$. Then the *definite integral* (定積分) *of* f *from* a *to* b is

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is *integrable* (可積) on $[a, b]$.

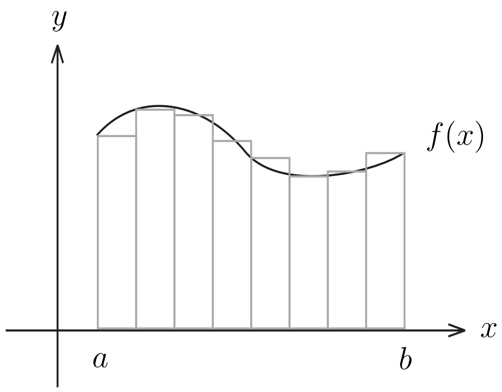

Figure 1: Definition of a definite integral.

The precise meaning of the limit that defines the integral is as follows:

> For every number $\varepsilon > 0$, there is an integer N such that
>
> $$\left| \int_a^b f(x)\,\mathrm{d}x - \sum_{i=1}^{n} f(x_i^*)\Delta x \right| < \varepsilon$$
>
> for every integer $n > N$ and for every choice of x_i^* in $[x_{i-1}, x_i]$.

There are some notations we should know:

$$\int_a^b f(x)\,\mathrm{d}x$$

integral sign:
integrand:
limits of integration:
lower limit (下限):
upper limit (上限):

☐ The procedure of calculating an integral is called *integration*.

☐ The $\mathrm{d}x$ simply indicates that the independent variable is x. (dummy variable)

☐ The definite integral $\int_a^b f(x)\,\mathrm{d}x$ is a number; it does not depend on x.

☐ The sum $\sum\limits_{i=1}^{n} f(x_i^*)\Delta x$ is called a *Riemann sum*.

☐ The geometric meaning of $\int_a^b f(x)\,\mathrm{d}x$ is the *net area* of $y = f(x)$ from a to b.

☐ In fact, the subinterval widths are not necessary equal width.

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

☐ Not all functions are integrable. For example, the Dirichlet function, or $f(x) = \frac{1}{x}$ on $0 < x \leq 1$.

Example 1. Show that $f(x) = \frac{1}{x}$ on $0 < x \leq 1$ is not integrable.

Solution.

Theorem 2 (page 380)**.** *If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuous, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,\mathrm{d}x$ exists.*

If f is integrable on $[a, b]$, then the limit of Riemann sum exists and gives the same value no matter how we choose the sample points x_i^*. To simplify the calculation of the integral, we often taken the sample points to be right endpoints. Then $x_i^* = x_i$ and the definition of an integral simplifies as follows.

Theorem 3 (page 380)**.** *If f is integrable on $[a, b]$, then*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Example 4 (page 383)**.** Set up an expression for $\int_1^3 e^x \,dx$ as a limit of sums.

Solution.

Example 5. Change the following limits of sums as integrals:

$$\lim_{n\to\infty} n^2 \left(\frac{1}{n^3} + \frac{1}{(n+1)^3} + \cdots + \frac{1}{(n+(2n-1))^3} \right)$$

Solution.

Evaluating Integrals

Example 6 (page 383)**.** Evaluate the expression in **Example 4**.

Solution.

☐ **Example 1** in section 5.1 is also an evaluating integral of $\int_0^1 x^2 \,dx$.

Properties of the Definite Integral

Properties of the Integral (page 385–387)**.**

(1) $\displaystyle\int_a^b c\,\mathrm{d}x = c(b-a)$, *where c is any constant.*

(2) $\displaystyle\int_a^b (f(x)+g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x.$

(3) $\displaystyle\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$, *where c is any constant.*

(4) $\displaystyle\int_a^b (f(x)-g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x.$

(5) $\displaystyle\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x.$

(6) *If $f(x) \geq 0$ for $a \leq x \leq b$, then* $\displaystyle\int_a^b f(x)\,\mathrm{d}x \geq 0.$

(7) *If $f(x) \geq g(x)$ for $a \leq x \leq b$, then* $\displaystyle\int_a^b f(x)\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x.$

(8) *If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then* $\displaystyle m(b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M(b-a).$

Example 7 (page 391)**.** If f is continuous on $[a,b]$, show that

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x.$$

Solution.