4.7 Optimization Problems (page 330)

Steps in solving optimization problems

1. **Understand the problem**: What is the unknown? What are the given quantities? What are the given conditions?

2. **Draw a diagram**: In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.

3. **Introduce notation**: Assign a symbol to the quantity that is to be maximized or minimized (call it Q for now). Also select symbols $a, b, c, \ldots, x, y$ for other known quantities and label the diagram with these symbols.

4. Express Q in terms of some of the other symbols.

5. If Q has been expressed as a function of more than one variable, use the given information to find relationships among these variables. Then use these equations to eliminate all but one of the variables. Thus we get $Q = f(x)$.

6. Use the methods of Section 4.1 and 4.3 to find the absolute maximum or minimum value of f.

Example 1 (Snell's Law, 斯乃爾定律, page 268)**.** Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to Fermat's Principle, a ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimizes the time taken. Show that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2},$$

where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are known.

Solution.

Example 2 (搬家俱, page 268)**.** A steel pipe is being carried down a hallway a m wide. At the end of the hall there is a right-angled turn into a narrower hallway b m wide. What is the length of the longest pipe that can be carried horizontally around the corner?

Solution.

Example 3 (看畫、教室的風水, page 269)**.** A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer. How far from the wall should the observer stand to get the best view? (In other words, where should the observer stand so as to maximize the angle θ subtended at his eye by the painting?)

Solution.

Example 4. A right circular cone is inscribed in a sphere of radius r. Find the largest possible volume of such a cone. In this case, what is the height and radius of the cone?

Solution.

Example 5 (折紙問題, page 269)**.** The upper right-hand corner of a piece of paper, 30 cm by 20 cm, is folded over to the bottom edge. How would you fold it so as to minimize the length of the fold? In other words, how would you choose x to minimize y?

Solution.

Example 6.

(a) Find the point (denote P) on the line $y = x^2$ that is closest to the point $Q(3, 0)$.

(b) Show that the line PQ is orthogonal to the tangent line of $y = x^2$ at P.

Solution.

Example 7 (電影的極佳位置)**.**