4.5 Summary of Curve Sketching (page 315)

Guidelines for sketching a curve

定 **Domain**: the set of x for which $f(x)$ is defined.

交 **Intercepts**: y-intercept $f(0)$, x-intercepts: let $y = 0$ and solve for x.

對 **Symmetry**: even function, odd function, periodic function.

漸 **Asymptotes**: horizontal asymptotes, vertical asymptotes, slant asymptotes.

一 **Intervals of increase or decrease**: use the Increasing/Decreasing test.

極 **Local maximum and minimum values**: find the critical numbers of f ($f'(c) = 0$ or $f'(c)$ does not exist.)

二 **Concavity and points of inflection**: compute $f''(x)$ and use the Concavity Test.

圖 **Sketch the Curve**: use the information in items 1–7, draw the graph.

Definition 1 (page 320)**.** If

$$\lim_{x\to\infty} (f(x) - (mx + b)) = 0,$$

where $m \neq 0$, then the line $y = mx + b$ is called a *slant asymptote* (斜漸近線).

Proposition 1. *The graph of $f(x)$ has a slant asymptote if and only if*

$$\lim_{x\to\infty} \frac{f(x)}{x} = m \neq 0 \quad \text{and} \quad \lim_{x\to\infty} (f(x) - mx) = b.$$

Proof. When $x > 0$,

$$\frac{f(x)}{x} = \frac{f(x) - (mx + b)}{x} + m + \frac{b}{x} \Rightarrow \lim_{x\to\infty} \frac{f(x)}{x} = 0 + m + 0 = m.$$

$$\begin{aligned} \lim_{x\to\infty} (f(x) - mx) &= \lim_{x\to\infty} (f(x) - (mx + b) + b) \\ &= \lim_{x\to\infty} (f(x) - (mx + b)) + \lim_{x\to\infty} b = 0 + b = b. \end{aligned}$$

Conversely, we have

$$\begin{aligned} \lim_{x\to\infty} (f(x) - (mx + b)) &= \lim_{x\to\infty} ((f(x) - mx) - b) \\ &= \lim_{x\to\infty} (f(x) - mx) - \lim_{x\to\infty} b = b - b = 0. \end{aligned}$$

□

☐ 函數圖形當 $x \to -\infty$ 也可能存在斜漸近線, 定義與其等價條件都改成 $\lim_{x\to-\infty}$。

☐ 兩個極限都要存在才能稱函數有斜漸近線。例如 $f(x) = \ln x$ 沒有斜漸近線。

Example 1 (page 317). Sketch the curve $y = \frac{2x^2}{x^2-1}$.

Solution.

A. The domain is ________________________.

B. The x- and y-intercept are both _.

C. Since ____________, the function f is ____.

D. Since

$$\lim_{x\to\pm\infty} \frac{2x^2}{x^2-1} = \underline{\hspace{4cm}},$$

the line _____ is a ____________________. The denominator is 0 when ______. we compute the following limits:

$$\lim_{x\to 1^+} \frac{2x^2}{x^2-1} = \qquad \lim_{x\to 1^-} \frac{2x^2}{x^2-1} =$$
$$\lim_{x\to -1^+} \frac{2x^2}{x^2-1} = \qquad \lim_{x\to -1^-} \frac{2x^2}{x^2-1} =$$

Therefore the lines _____ and ______ are vertical asymptotes.

E. Direct computation gives

$$y' = \underline{\hspace{8cm}}.$$

Since $f'(x) > 0$ when ________________ and $f'(x) < 0$ when _______________, f is increasing on __________________ and decreasing on _______________.

F. The only critical number is _____. Since f' changes from positive to negative at 0, $f(0) = 0$ is a ______________ by the First Derivative Test.

G. Direct computation gives

$$f''(x) = \underline{\hspace{10cm}}.$$

We know $f''(x) > 0$ on ______ and $f''(x) < 0$ on ______. Thus the curve is concave upward on the interval ___________________ and concave downward on ______. It has no point of inflection since _________________________________.

H. Using this information to sketch the curve. (畫在右上角)

Example 2 (page 317)**.** Sketch the curve $y = \frac{x^2}{\sqrt{x+1}}$.

Solution.

A. The domain is _______.

B. The x- and y-intercept are both _.

C. Symmetry: None.

D. Since

$$\lim_{x\to\infty} \frac{x^2}{\sqrt{x+1}} = \_\_,$$

there is no horizontal asymptote. Since

$$\lim_{x\to -1^+} \frac{x^2}{\sqrt{x+1}} = \_\_,$$

the line ______ is a vertical asymptotes.

E. Direct computation gives

$$y' = \frac{\qquad\qquad\qquad}{\qquad\qquad\qquad}.$$

We see that $f'(x) = 0$ when _____, so the only critical number is _. Since $f'(x) > 0$ when _____ and $f'(x) < 0$ when __________, f is increasing on ______ and decreasing on ______.

F. Since $f'(0) = 0$ and f' changes from negative to positive at 0, $f(0) = 0$ is a __________________________ by the First Derivative Test.

G. Direct computation gives

$$f''(x) = \frac{\qquad\qquad\qquad}{\qquad\qquad\qquad}.$$

Since the numerator is always _______, we know $f''(x) > 0$ for all x in the domain of f, which means f is concave upward on _______ and there is no point of inflection.

H. Using this information to sketch the curve. (畫在右上角)

Example 3 (page 318)**.** Sketch the curve $y = xe^x$.

Solution.

A. The domain is _.

B. The x- and y-intercept are both _.

C. Symmetry: None.

D. Since

$$\lim_{x \to \infty} xe^x = \_\_,$$

there is no horizontal asymptote. By the l'Hospital Rule, we have

$$\lim_{x \to -\infty} xe^x = \lim_{x \to -\infty} \frac{x}{e^{-x}} = \underline{\hspace{6em}},$$

so the _____ is a horizontal asymptote.

E. Direct computation gives

$$y' = \underline{\hspace{8em}}.$$

Since $f'(x) > 0$ when ______ and $f'(x) < 0$ when ______, f is increasing on ______ and decreasing on _______.

F. Since $f'(-1) = 0$ and f' changes from negative to positive at $x = -1$, $f(-1) = -e^{-1}$ is a ________________________ by the First Derivative Test.

G. Direct computation gives

$$f''(x) = \underline{\hspace{10em}}.$$

Since $f''(x) > 0$ if ______ and $f''(x) < 0$ if ______, f is concave upward on ______ and concave downward on ________. The inflection point is _________.

H. Using this information to sketch the curve. (畫在右上角)

Example 4 (page 319). Sketch the curve $y = \frac{\cos x}{2+\sin x}$.

Solution.

A. The domain is __.

B. The x-intercepts are ________________ and y-intercept is ________.

C. Symmetry: f is neither even nor odd. Since $f(x + 2\pi) = f(x)$ for all x, f is ________ and has period __. Thus, the following steps we only consider $0 \le x \le 2\pi$ and then extend the curve by translation.

D. Asymptotes: None.

E. Direct computation gives

$$y' = \frac{\qquad\qquad}{\qquad\qquad}.$$

Thus $f'(x) > 0$ when ________________________________. So f is increasing on ________ and decreasing on ________________.

F. From part **E** and First Derivative Test, we see that the local minimum value is ____________ and local maximum value is ____________.

G. Direct computation gives

$$f''(x) = \frac{\qquad}{\qquad}.$$

Since $f''(x) > 0$ if __________, f is concave upward on ______ and concave downward on ________________. The inflection point is ________________.

H. Using this information to sketch the curve.

Example 5 (page 319)**.** Sketch the curve $y = \ln(4 - x^2)$.

Solution.

A. The domain is ______.

B. The y-intercept is $f(0) = \ln 4$. To find the x-intercept, we set $\ln(4 - x^2) = 0$, so we have ____________________. Therefore the x-intercepts are ____.

C. Since $f(-x) = f(x)$, f is ____ and the curve is symmetric about the _____.

D. Since

$$\lim_{x \to -2^+} \ln(4 - x^2) = \underline{\hspace{2em}}, \qquad \lim_{x \to 2^-} \ln(4 - x^2) = \underline{\hspace{2em}},$$

the lines ________________ are vertical asymptotes.

E. Direct computation gives

$$y' = \underline{\hspace{3em}}.$$

Since $f'(x) > 0$ when __________ and $f'(x) < 0$ when _________, f is increasing on ______ and decreasing on ____.

F. The only critical number is _____. Since f' changes from positive to negative at 0, $f(0) = \ln 4$ is a ___________________________ by the First Derivative Test.

G. Direct computation gives

$$f''(x) = \underline{\hspace{16em}}.$$

Since $f''(x) < 0$ for all x, the curve is ________________ on ______ and has no inflection point.

H. Using this information to sketch the curve.

Example 6 (page 320). Sketch the curve $y = \frac{x^3}{x^2+1}$.

Solution.

A. The domain is _.

B. The x- and y-intercept are both _.

C. Since ______________, the function f is ____.

D. Since $x^2 + 1$ is never 0, there is no vertical asymptote. Since $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, there is no horizontal asymptote. Long division gives

$$f(x) = \frac{x^3}{x^2+1} = \underline{\hspace{3cm}},$$

$$f(x) - x = -\frac{x}{x^2+1} = \underline{\hspace{7cm}}.$$

So the line _____ is a ______________.

E. Direct computation gives

$$y' = \underline{\hspace{9cm}}.$$

Since $f'(x) > 0$ when _________________, f is increasing on ________.

F. Although $f'(0) = 0$, f' does not change sign at 0, so there is _________________ or _________.

G. Direct computation gives

$$f''(x) = \underline{\hspace{15cm}}.$$

Since $f''(x) = 0$ when _______________, we set up the following chart.

The points of inflection are ____________________________.

H. Using this information to sketch the curve.