4.4 Indeterminate Forms and l'Hospital's Rule (page 304)

In this section, we want to introduce a new method to deal with the limit such as

$$\lim_{x\to\infty} \frac{\ln x}{x-1}, \quad \text{or} \quad \lim_{x\to\infty} \frac{x^2}{\mathrm{e}^x}.$$

Definition 1 (page 304–305)**.**

(1) If we have a limit of the form $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$, where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, it is called an *indeterminate form of type* $\frac{0}{0}$ (零分之零的不定型).

(2) If we have a limit of the form $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$, where both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$) as $x \to a$, it is called an *indeterminate form of type* $\frac{\infty}{\infty}$ (無限大分之無限大的不定型).

L' Hospital's Rule (page 305)**.** *Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that a limit has an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

☐ 務必檢查定理的條件: (1) 是否爲不定型; (2) 檢查 $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ 是否存在。

☐ 定理也適用於單邊極限。

☐ 可以串聯至有限次可微分函數。$\lim\limits_{x\to a} \frac{f(x)}{g(x)} \overset{L}{=} \lim\limits_{x\to a} \frac{f'(x)}{g'(x)} \overset{L}{=} \cdots \overset{L}{=} \lim\limits_{x\to a} \frac{f^{(k)}(x)}{g^{(k)}(x)} = M.$

Example 2.

(a) Find $\lim\limits_{t\to 0^+} \frac{t-\ln(1+t)}{t^2}$.

(b) Use (a) to find $\lim\limits_{t\to 0^+} \frac{\sqrt{t-\ln(1+t)}}{t}$.

Solution.

Indeterminate Products, page 308

Definition 3 (page 305)**.** If we have a limit of the form $\lim_{x\to a} f(x)g(x)$, where $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$) as $x \to a$, it is called an *indeterminate form of type* $0 \cdot \infty$. (零乘以無限大的不定型)

We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f},$$

and this converts the given limit into an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 4 (page 308)**.** Evaluate $\lim_{x\to 0^+} x \ln x$.

Solution.

☐ 如何把函數分配至分子或分母是一門學問。

Indeterminate Differences, page 309

Definition 5 (page 305)**.** If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a}(f(x) - g(x))$$

is called an *indeterminate form of type* $\infty - \infty$ (無限大減無限大的不定型).

We can try to convert the difference into a quotient (for instance, by using a common denominator (通分), or rationalization (有理化), or factoring out a common factor (提公因式)) so that we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 6. Find the limit $\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$.

Solution.

☐ 隨時觀察並恰當整理函數, 不要盲目使用 l'Hospital Rule。

Indeterminate Powers, page 310

Definition 7 (page 310)**.** Several indeterminate forms arise from the limit

$$\lim_{x\to a}(f(x))^{g(x)}$$

(1) $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$: type 0^0.

(2) $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$: type ∞^0.

(3) $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$: type 1^∞.

Each of these three cases can be treated either by taking the natural logarithm: let $y = (f(x))^{g(x)}$, then $\ln y = g(x)\ln f(x)$ or by writing the function as an exponential: $(f(x))^{g(x)} = e^{g(x)\ln f(x)}$. In either method we are led to the indeterminate product $g(x)\ln f(x)$, which is of type $0\cdot\infty$.

Example 8 (page 310)**.** Find $\lim_{x\to 0^+} x^x$.

Solution.

☐ 取對數算出極限值後, 記得還原。

幾個使用 **l'Hospital Rule** 的經驗

- 使用前務必檢查定理的條件: (1) 是否為不定型; (2) 檢查 $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ 是否存在。

$$\lim_{x\to 0}\frac{x}{1+\sin x}$$

$$\lim_{x\to \infty}\frac{x-\sin x}{x+\sin x}$$

- 避免鬼打牆; 像是 $\sin x, \cos x$ (as $x\to\infty$) 或是 $\sin\frac{1}{x}, \cos\frac{1}{x}, \frac{1}{x}, \frac{1}{\ln x}$ (as $x\to 0$)。

$$\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x} \overset{(\frac{0}{0}),L}{=} \lim_{x\to 0}\frac{2x\sin\frac{1}{x}+x^2\cos\frac{1}{x}(-\frac{1}{x^2})}{\cos x} = \lim_{x\to 0}\frac{2x\sin\frac{1}{x}-\cos\frac{1}{x}}{\cos x}\ldots ?$$

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{x}{\frac{1}{\ln x}} \overset{(\frac{0}{0}),L'}{=} \lim_{x\to 0^+}\frac{1}{-\frac{1}{(\ln x)^2}\cdot\frac{1}{x}} = \lim_{x\to 0^+} -x(\ln x)^2 \ldots \text{ 鬼打牆}$$

- 分子分母適時地整理、重新分配, 或是變數變換, 有助於計算。

$$\lim_{x\to 0^+}\frac{1}{x^2 e^{\frac{1}{x}}} = \lim_{x\to 0^+}\frac{e^{-\frac{1}{x}}}{x^2} \overset{(\frac{0}{0}),L'}{=} \lim_{x\to 0^+}\frac{e^{-\frac{1}{x}}\cdot\frac{1}{x^2}}{2x} = \lim_{x\to 0^+}\frac{e^{-\frac{1}{x}}}{2x^3}\ldots \text{ 鬼打牆}$$

$$\lim_{x\to 0^+}\frac{1}{x^2 e^{\frac{1}{x}}} = \lim_{x\to 0^+}\frac{\frac{1}{x^2}}{e^{\frac{1}{x}}} \overset{(\frac{\infty}{\infty}),L}{=} \lim_{x\to 0^+}\frac{-2\frac{1}{x^3}}{e^{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0^+}\frac{2}{x^5 e^{\frac{1}{x}}}\ldots \text{ 鬼打牆}$$

$$\lim_{x\to 0^+}\frac{1}{x^2 e^{\frac{1}{x}}}$$

$$\lim_{x\to 0}\frac{\tan x - x}{x-\sin x}$$

$$\lim_{x\to 0^+}\frac{\tan 3x}{\sqrt{1-\cos 2x}}$$

- l'Hospital Rule 某種程度是“大絕”, 但並非萬能。

$$\lim_{x\to\infty}\frac{(\sin x)e^x}{(x+\sin x)e^{2x}}$$

- 記得其他求極限的方法, 像是 Squeeze Theorem, definition of derivative, 也很好用。(之後還會介紹用積分方法求極限, 下學期會介紹使用泰勒展式法求極限。)

$$\lim_{x\to 0}\frac{\sin x}{x} = \_\qquad \lim_{x\to\infty} x\sin\frac{1}{x} = \_\qquad \lim_{x\to\infty}\frac{\sin x}{x} = \_\qquad \lim_{x\to 0} x\sin\frac{1}{x} = \_$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \_\qquad \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \_\qquad \lim_{x\to 0}\frac{\ln(1+x)}{x} = \_$$

$$\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x = \_\qquad \lim_{x\to\infty}(1+x)^{\frac{1}{x}} = \_\qquad \lim_{x\to\infty}\frac{\ln(1+x)}{x} = \_$$

Appendix

Proof of l'Hospital's Rule (Appendix A46)

We are assuming that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. Let $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$. Define

$$F(x) = \begin{cases} f(x) & \text{if } x \neq a \\ 0 & \text{if } x = 1 \end{cases}, \qquad G(x) = \begin{cases} g(x) & \text{if } x \neq a \\ 0 & \text{if } x = 1 \end{cases}.$$

Then <u>both F and G are continuous on I</u> since $f$and g are continuous on $\{x \in I | x \neq a\}$ and

$$\lim_{x\to a} F(x) = \lim_{x\to a} f(x) = 0 = F(a), \qquad \lim_{x\to a} G(x) = \lim_{x\to a} g(x) = 0 = G(a).$$

Furthermore, <u>F and G are differentiable on (a, x) (or (x, a))</u> since $F' = f'$ and $G' = g'$. Since $G' \neq 0$, by the <u>Cauchy's Mean Value Theorem</u>, there is a number y such that $a < y < x$ (or $x < y < a$) and

$$\frac{F'(y)}{G'(y)} = \frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F(x)}{G(x)}.$$

Hence

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{F(x)}{G(x)} = \lim_{y\to a^+} \frac{F'(y)}{G'(y)} = \lim_{y\to a^+} \frac{f'(y)}{g'(y)} = L,$$
$$\left(\text{and} \quad \lim_{x\to a^-} \frac{f(x)}{g(x)} = \lim_{x\to a^-} \frac{F(x)}{G(x)} = \lim_{y\to a^-} \frac{F'(y)}{G'(y)} = \lim_{y\to a^-} \frac{f'(y)}{g'(y)} = L.\right)$$

Therefore,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

This proves l'Hospital's Rule for the case where <u>a is finite</u>.

<u>If a is infinite</u>, we let $t = \frac{1}{x}$. Then $t \to 0^+$ as $x \to \infty$, so we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0^+} \frac{f(\frac{1}{t})}{g(\frac{1}{t})} \overset{L}{=} \lim_{t\to 0^+} \frac{f'(\frac{1}{t}) \cdot \left(-\frac{1}{t^2}\right)}{g'(\frac{1}{t}) \cdot \left(-\frac{1}{t^2}\right)} = \lim_{t\to 0^+} \frac{f'(\frac{1}{t})}{g'(\frac{1}{t})} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

L'Hospital Rule 與極限法則的合併使用

回想極限的四則運算法則與羅必達法則:

Limit Laws (page 99)**.** *Suppose that* $\lim_{x\to a} p(x)$ *and* $\lim_{x\to a} q(x)$ *exist. Then*

$$\lim_{x\to a}\frac{p(x)}{q(x)}=\frac{\lim_{x\to a} p(x)}{\lim_{x\to a} q(x)} \quad \text{if} \quad \lim_{x\to a} q(x)\neq 0.$$

L' Hospital's Rule (page 302)**.** *Suppose* f *and* g *are differentiable and* $g'(x)\neq 0$ *on an open interval* I *that contains* a (*except possibly at* a). *Suppose that*

$$\lim_{x\to a} f(x)=0 \quad \textit{and} \quad \lim_{x\to a} g(x)=0$$

or that

$$\lim_{x\to a} f(x)=\pm\infty \quad \textit{and} \quad \lim_{x\to a} g(x)=\pm\infty$$

(*In other words, we have an indeterminate form of type* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$.) *Then*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (*or is* ∞ *or* $-\infty$).

這兩個定理可以搭配起來靈活運用, 比方說:

Theorem 9. *Suppose that* $p(x), q(x)$ *satisfy the assumptions of Limit Laws with* $\underline{\lim_{x\to a} p(x)\neq 0}$, *and suppose* $f(x), g(x)$ *satisfy the assumptions of L'Hospital Rule. Then*

$$\lim_{x\to a}\frac{p(x)f(x)}{q(x)g(x)}=\lim_{x\to a}\frac{p(x)}{q(x)}\cdot\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

上述定理要強調的是說: 雖然分子 $p(x)f(x)$ 整體看取極限是 0, 分母 $q(x)g(x)$ 整體看取極限是 0, 於是 $\frac{p(x)f(x)}{q(x)g(x)}$ 是 $\frac{0}{0}$ 的不定型, 但是當分子與分母各自只有一部分是不定型 (只有 $\frac{f}{g}$ 是 $\frac{0}{0}$), 而 <u>p, q 具有非零的極限</u>, 那麼就可以把 p, q 的極限抽出來, 考慮剩下的不定型之極限, 再相乘。

理由很簡單, 因爲 $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=L$, 所以下式的第一個等式合法 (乘法法則)

$$\lim_{x\to a}\frac{p(x)f(x)}{q(x)g(x)}=\lim_{x\to a}\frac{p(x)}{q(x)}\cdot\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{p(x)}{q(x)}\cdot\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

定理一就是大大化簡計算的一個方法, 因爲有很多人一看到 $\frac{0}{0}$ 就不加思索的地上下微分(典型的“看見黑影就開槍”), 雖然還是可以算出答案, 但是過程中可能會添上很多計算上的麻煩。