4.3 How Derivatives Affect the Shape of a Graph (page 293)

Increasing/Decreasing Test (page 293).

(a) *If $f'(x) > 0$ on an interval, then f is increasing on that interval.*

(b) *If $f'(x) < 0$ on an interval, then f is decreasing on that interval.*

Proof.

(a) Let $x_1 < x_2$. By the ____________________, there is $c \in (x_1, x_2)$ such that

.

(b) Let $x_1 < x_2$. By the ____________________, there is $c \in (x_1, x_2)$ such that

. □

Example 1. Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Solution. We compute $f'(x) =$

Solutions of $f'(x) = 0$ are ____________. Hence

$f(x)$ is increasing on ________________; $f(x)$ is decreasing on _________________.

The First Derivative Test (page 294). *Suppose that c is a critical number of a continuous function f.*

(a) *If f' changes from positive to negative at c, then f has a local maximum at c.*

(b) *If f' changes from negative to positive at c, then f has a local minimum at c.*

(c) *If f' does not change sign at c (for example, if f' is positive on both side of c or negative on both sides), then f has no local maximum or minimum at c.*

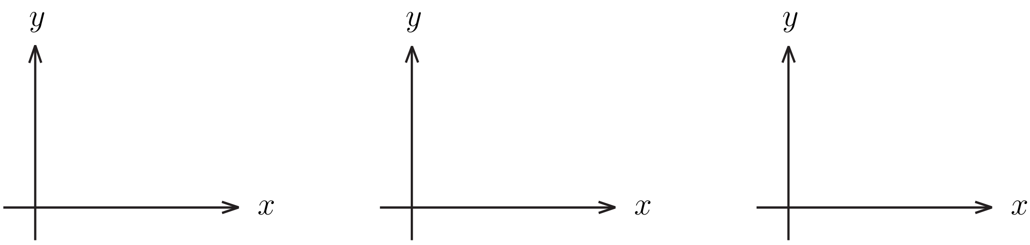

Figure 1: The First Derivative Test.

Example 2. Find the local minimum and maximum values of the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ in **Example 1**.

Solution.

x	-1	0	2
f	0	5	-27
f'			

Hence f has local maximum ________; f has local minimum ____________________.

Definition 3 (page 296)**.** If the graph f lies above all of it tangents on an interval I, then it is called *concave upward* (凹口朝上) on I. If the graph f lies below all of it tangents on an interval I, then it is called *concave downward* (凹口朝下) on I.

☐ 有些教科書或文獻使用凸函數 (convex) 取代凹口向上 (concave upward)。

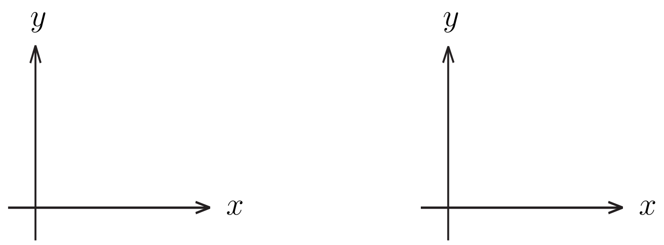

Figure 2: Concave upward and concave downward.

Concavity Test (page 296)**.**

(a) *If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.*

(b) *If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.*

Proof of (a). Since $f''(x) > 0$ in I, we know that $f'(x)$ is increasing in I. Given $x_0 \in I$, the tangent line equation to the graph of $f(x)$ at $(x_0, f(x_0))$ is

$$y - f(x_0) = f'(x_0)(x - x_0) \Rightarrow y = f'(x_0)(x - x_0) + f(x_0).$$

We will show that $f(x) \geq f'(x_0)(x - x_0) + f(x_0)$ for all $x \in I$.

Consider the function

$$F(x) = f(x) - f'(x_0)(x - x_0) - f(x_0) \quad \text{for} \quad x \in I.$$

First, we know that $F(x_0) = 0$. Next, we compute $F'(x) = f'(x) - f'(x_0)$, which implies $F'(x_0) = f'(x_0) - f'(x_0) = 0$. Since $F'(x) < 0$ for $x < x_0$ and $F'(x) > 0$ for $x > x_0$, we know that $F(x_0)$ is a local (and hence absolute) minimum at $x = x_0$ in I. That means $F(x) \geq 0$ for all $x \in I$, thus $f(x) \geq f'(x_0)(x - x_0) + f(x_0)$ for all $x \in I$. ☐

Definition 4 (page 297)**.** A point P on a curve $y = f(x)$ is called an *inflection point* (反曲點) if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

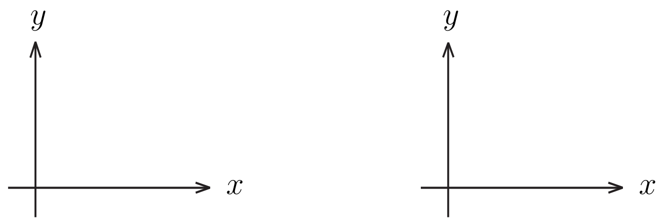

Figure 3: Inflection points.

Example 5. Find the concave upward and downward intervals, and inflection points of the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ in **Example 1**. Sketch the graph of f.

Solution. We compute

$$f''(x) = \qquad\qquad\qquad\qquad .$$

So

x		-1	x_1	0	x_2	2	
f		0	$f(x_1)$	5	$f(x_2)$	-27	
f'	$-$	0	$+$	0	$-$	0	$+$
f''							

The points of inflections are ____________________.

f is concave upward on ____________________.

f is concave downward on __________.

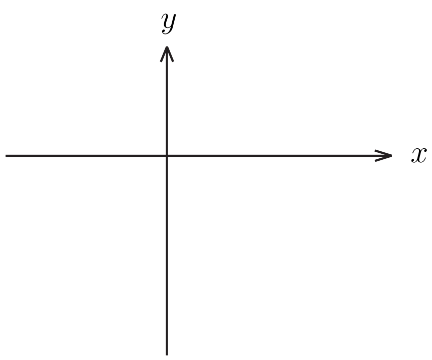

Figure 4: The graph of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

The Second Derivative Test (page 297)**.** *Suppose f'' is continuous near c.*

(a) *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.*

(b) *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.*

Example 6. Show that $f(x) = \frac{\sin x}{x}$ is decreasing on $(0, \frac{\pi}{2})$.

Solution.

☐ 比較 Section 2.3, 那時候證明了 $|\sin x| \le |x|$。

Example 7. Classify all cubic functions $f(x) = ax^3 + bx^2 + cx + d$.

Solution.