4.2 The Mean Value Theorem (page 287)

Question 1. A highway from Taipei to Kaohsiung is 330 km and the speed limit is 110 km/h. Man A drove the car on the high way from Taipei at 9 : 00 AM to Kaohsiung at 11 : 59 AM. Did he exceed the speed limit?

Theorem 2 (Rolle's Theorem, page 287)**.** *Let f be a function that satisfies the following three hypotheses:*

(1) *f is continuous on the closed interval $[a, b]$.*

(2) *f is differentiable on the open interval (a, b).*

(3) *$f(a) = f(b)$.*

Then there is a number c in (a, b) such that $f'(c) = 0$.

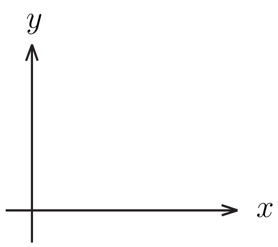

Figure 1: Rolle's Theorem.

Proof. There are three cases.

(I) <u>$f(x) = k$, a constant.</u> We have $f'(x) = 0$, so the number c can be taken to be *any* number in (a, b).

(II) <u>$f(x) > f(a)$ for some x in (a, b).</u> By the ________________________, f has a maximum somewhere in $[a, b]$. Since $f(a) = f(b)$, it must attain this maximum value at a number c in the open interval (a, b). Then f has a ______________ at c, and f is differentiable at c. By _________________, we know $f'(c) = 0$.

(III) <u>$f(x) < f(a)$ for some x in (a, b).</u> By the ________________________, f has a minimum value in $[a, b]$, and since $f(a) = f(b)$, it attains this local minimum value at a number $c \in (a, b)$. By _________________, $f'(c) = 0$.

□

☐ 定理條件, 函數 $f(x)$ 必須在「閉區間連續」。

☐ 定理條件, 函數 $f(x)$ 必須在開區間「可微分」(每一個點)。

☐ 定理結論只告知「存在性」。

Example 3. Give examples that each condition in Rolle's Theorem is required.

Solution.

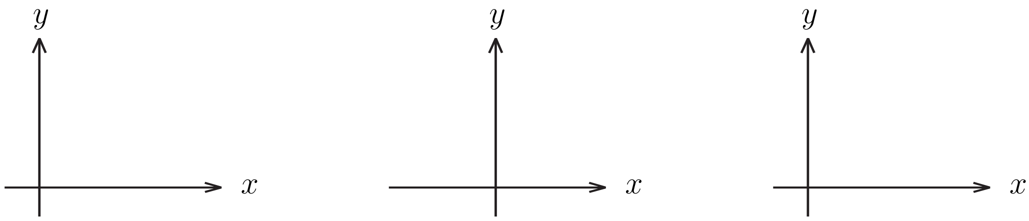

Figure 2: Study Rolle's Theorem.

Example 4 (page 287)**.** Prove that $x^3 + x - 1 = 0$ has exactly one real root.

Solution.

Theorem 5 (The Mean Value Theorem, 平均值定理, page 288)**.** *Let f be a function that satisfies the following hypotheses:*

(1) *f is continuous on the closed interval $[a, b]$.*

(2) *f is differentiable on the open interval (a, b).*

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \textit{or equivalently,} \quad f(b) - f(a) = f'(c)(b - a).$$

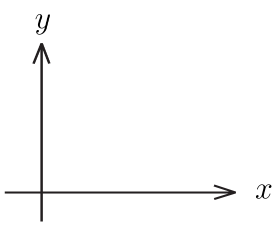

Figure 3: The Mean Value Theorem.

Proof of Mean Value Theorem. Define a new function

$$h(x) =$$

We will verify that $h(x)$ satisfies the three hypotheses of Rolle's Theorem.

(1) The function h is continuous on $[a, b]$: It is the sum of f and a first-degree polynomial, both of which are continuous.

(2) The function h is differentiable on (a, b): Both f and the first-degree polynomial are differentiable. In fact, we have

$$h'(x) =$$

(3) $h(a) = h(b) = 0$:

$$h(a) =$$

$$h(b) =$$

By ______________, there is a number $c \in (a, b)$ such that $h'(c) = 0$. Therefore,

□

☐ 均值定理也是要注意「連續」、「可微」、「存在」。

☐ 爲什麼這個定理要叫做「平均值定理」?

Theorem 6. *If* $f'(x) = 0$ *for all* x *in an interval* (a, b)*, then* f *is constant on* (a, b)*.*

Proof. Let x_1 and x_2 be any two numbers in (a, b) with $x_1 < x_2$. Since f is differentiable on (a, b), it must be differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$. By applying the ___________________ to f on the interval $[x_1, x_2]$, we get a number c such that $x_1 < c < x_2$ and

Therefore f has the same value at *any* two numbers x_1 and x_2 in (a, b). So $f(x)$ is constant on (a, b). □

☐ **Theorem 6** 提供一個刻劃常數函數的方法。

Corollary 7. *If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.*

Proof. Let $F(x)$

□

□ 考慮 $f(x) = \frac{x}{|x|}$。

Example 8. Show that $\left|\tan\frac{x}{2} - \tan\frac{y}{2}\right| \geq \frac{|x-y|}{2}$ for any $x, y \in (-\pi, \pi)$.

Solution. If $x = y$, the inequality holds. If $x \neq y$, without loss of generality, we assume $-\pi < x < y < \pi$. Consider the function $f(t) = \tan\frac{t}{2}$, then

- $f(t)$ is ________________
- $f(t)$ is __________________

By the __________________, there is a number $c \in (x, y)$ such that $f(x) - f(y) = f'(c)(x - y)$, which implies $|f(x) - f(y)| = |f'(c)||x - y|$. Since $f'(t) =$ ______, we have $|f'(c)| =$ _________. So $|f(x) - f(y)| \geq \frac{1}{2}|x - y|$, which means

$$\left|\tan\frac{x}{2} - \tan\frac{y}{2}\right| \geq \frac{|x-y|}{2}.$$

Theorem 9 (Cauchy's Mean Value Theorem, (柯西均值定理) Appendix F, A45)**.** *Suppose that the functions f and g are continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ for all x in (a, b). Then there is a number $c \in (a, b)$ such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. The key point is to find a new function $F(x)$ and apply the Mean Value Theorem.

$F(x) =$

□