3.11 Hyperbolic Functions (page 259)

Hyperbolic Functions, page 259

Certain combinations of e^x and e^{-x} arise so frequently in mathematics and engineering. In many ways they are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason they are called *hyperbolic functions* (雙曲函數) and individually called *hyperbolic sine, hyperbolic cosine*, and so on.

Definition 1 (Definition of the hyperbolic functions, page 259)**.**

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$
$$\coth x = \frac{\cosh x}{\sinh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x}.$$

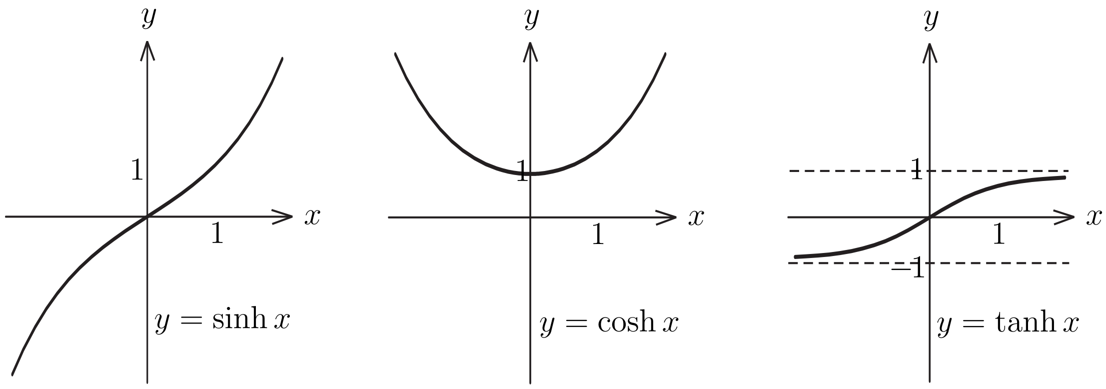

Figure 1: Hyperbolic functions.

Hyperbolic Identities (page 260)**.**

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$
$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

The identity $\cosh^2 x - \sinh^2 x = 1$ indicates the curve $(\cosh x, \sinh x)$ is hyperbola.

Derivatives of Hyperbolic Functions (page 261)**.**

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\cosh x = \sinh x$$
$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \qquad \frac{d}{dx}\coth x = -\operatorname{csch}^2 x$$
$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x \qquad \frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x$$

Inverse Hyperbolic Functions

Since the hyperbolic functions are defined in terms of exponential functions, the inverse hyperbolic functions can be expressed in terms of logarithms:

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$$
$$\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$$
$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Derivatives of Inverse Hyperbolic Functions (page 263)**.**

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\tanh^{-1} x = \frac{1}{1-x^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\coth^{-1} x = \frac{1}{1-x^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{sech}^{-1} x = -\frac{1}{x\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{csch}^{-1} x = -\frac{1}{|x|\sqrt{x^2+1}}$$

Example 2 (page 265)**.** Using principles from physics it can be shown that when a cable is hung between two poles, it takes the shape of a curve $y = f(x)$ that satisfies the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\rho g}{T}\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2},$$

where ρ is the linear density of the cable, g is the acceleration due to gravity, T is the tension in the cable at its lowest point, and the coordinate system is chosen appropriately.

The function

$$y = f(x) = \frac{T}{\rho g}\cosh\left(\frac{\rho g}{T}x\right)$$

is a solution of this differential equation.

☐ A curve with equation $y = c + a\cosh\left(\frac{x}{a}\right)$ is called a *catenary* (懸鏈線).

Example 3 (page 265)**.** Another application of hyperbolic functions occurs in the description of ocean waves: The velocity of a water wave with length L moving across a body of water with depth d is modeled by the function

$$v = \sqrt{\frac{gL}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)},$$

where g is the acceleration due to gravity.