3.10 Linear Approximations and Differentials (page 251)

Recall two high school mathematics questions.

Example 1. Let $f(x) = 3x^3 - 22x^2 + 54x - 43$. Find $f(2.001)$ correct to three decimal place.

Solution.

Example 2. Find 1.0001^{100} correct to two decimal place.

Solution.

☐ 如何抓出一個量的「主要部份」?

A curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line. This observation is the basis for a method of finding approximate values of functions.

The idea is that it might be easy to calculate a value $f(a)$ of a function, but difficult to compute nearby values of f. So we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$.

Given a curve $y = f(x)$, an equation of the tangent line at $(a, f(a))$ is

$$y = f(a) + f'(a)(x - a).$$

Definition 3 (page 252)**.** The approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the *linear approximation* or *tangent line approximation* of f at a (線性估計, 切線估計). The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the *linearization* of f at a (線性化).

Example 4 (page 252)**.** Approximate the numbers $\sqrt{3.98}$.

Solution.

☐ 選取「適合的點」進行線性估計。

Example 5. Using a linear approximation to estimate $\cot 46^\circ$.

Solution.

☐ 角的單位必須換成「弧度」再計算。

Example 6. Go back to **Example 1.** and **Example 2.** to find out the calculation is in fact the linear approximation.

Solution.

(1) $f(x) = 3x^3 - 22x^2 + 54x - 43 = 1 + 2(x-2) - 4(x-2)^2 + 3(x-2)^3$.

$f'(x) = 2 - 8(x-2) + 9(x-2)^2$. $f(2.001) \approx f(2) + f'(2)(2.001 - 2) = 1.002$.

(2) $f(x) = (1+x)^{100} = C_0^{100}1^{100}x^0 + C_1^{100}1^{99}x^1 + C_2^{100}1^{98}x^2 + \cdots + C_{100}^{100}1^0x^{100}$.

$f'(x) = 100(1+x)^{99}$. $f(0.0001) \approx f(0) + f'(0)(0.0001 - 0) = 1.01$.

Applications to Physics

Differentials

The ideas behind linear approximations are sometimes formulated in the terminology and notation of *differentials* (微分). If $y = f(x)$, where f is a differentiable function, then the *differential* $\mathrm{d}x$ is an independent variable; that is, $\mathrm{d}x$ can be given the value of any real number. The *differential* $\mathrm{d}y$ is then defined in terms of $\mathrm{d}x$ by the equation

$$\mathrm{d}y \overset{\text{def.}}{=} f'(x)\,\mathrm{d}x.$$

So $\mathrm{d}y$ is a dependent variable; it depends on the values of x and $\mathrm{d}x$.

The geometric meaning of differentials is shown in Figure 1.

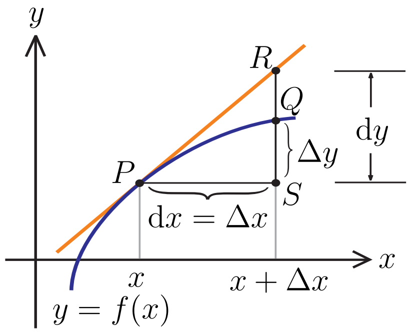

Figure 1: Geometric meaning of differentials.

Let $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$ be points on the graph of f and let $\mathrm{d}x = \Delta x$. The corresponding change in y is $\Delta y = f(x + \Delta x) - f(x)$. The slope of the tangent line PR is the derivative $f'(x)$. Thus the directed distance from S to R is $f'(x)\,\mathrm{d}x = \mathrm{d}y$. Therefore $\mathrm{d}y$ represents the amount that the tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount $\mathrm{d}x$.

Example 7 (page 254)**.** Compare the values of Δy and $\mathrm{d}y$ if $y = f(x) = x^3 + x^2 - 2x + 1 = 9 + 14(x - 2) + 7(x - 2)^2 + (x - 2)^3$ and x changes from 2 to 2.05.

Solution. Since $f(2) = 9$ and $f(2.05) = 9 + 14 \cdot 0.05 + 7 \cdot (0.05)^2 + (0.05)^3 = 9.717625$, we have

☐ 了解實際誤差 Δy 與微分 $\mathrm{d}y$ 的差別。

Example 8 (page 255)**.** The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere.

Solution.

☐ 看到這樣的誤差, 你對這個球形物的「設計」感到滿意嗎?

Although the possible error in Example 8 may appear to be rather large, a better picture of the error is given by the *relative error* (相對誤差), which is computed by dividing the error by the total volume:

$$\frac{\Delta V}{V} \sim \frac{\mathrm{d}V}{V} = \frac{4\pi r^2 \mathrm{d}r}{\frac{4}{3}\pi r^3} = 3\frac{\mathrm{d}r}{r}.$$

Thus the relative error in the volume is about three times the relative error in the radius. In Example 8 the relative error in the radius is approximately

$$\frac{\mathrm{d}r}{r} = \frac{0.05}{21} \sim 0.0024$$

and it produces a relative error of about 0.007 in the volume. The error could also be expressed as *percentage error* (誤差百分比) of 0.24% in the radius and 0.7% in the volume.