3.8 Exponential Growth and Decay (page 237)

In this section, we will show some examples of quantities grow or decay at a rate proportional to their size:

(1) The number of individuals in a population of animals or bacteria.

(2) In nuclear physics, the mass of a radioactive substance decays at a rate proportional to the mass.

(3) In chemistry, the rate of unimolecular first-order reaction is proportional to the concentration of the substance.

(4) In finance, the value of a savings account with continuously compounded interest increases at a rate proportional to that value.

Definition 1 (page 237)**.** If $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky, \tag{1}$$

where k is a constant. It is called the *law of natural growth* (if $k > 0$) or the *law of natural decay* (if $k < 0$). The equation (1) is called a *differential equation* (微分方程) because it involves an unknown function y and its derivative $\frac{\mathrm{d}y}{\mathrm{d}t}$.

Theorem 2 (page 237)**.** *The only solutions of the differential equation* $\frac{\mathrm{d}y}{\mathrm{d}t} = ky$ *are the exponential functions*

$$y(t) = y(0)\mathrm{e}^{kt}.$$

Proof. Here we check any exponential function of the form $y(t) = C\mathrm{e}^{kt}$, where C is a constant, satisfies

$$y'(t) = C(k\mathrm{e}^{kt}) = k(C\mathrm{e}^{kt}) = ky(t).$$

We will prove in section 9.4 that *any* function that satisfies $\frac{\mathrm{d}y}{\mathrm{d}t} = ky$ must be of the form $y = C\mathrm{e}^{kt}$.

To see the significance of the constant C, we observe that

$$y(0) = C\mathrm{e}^{k\cdot 0} = C.$$

Therefore C is the initial value of the function. □

□ 唯有指數函數滿足微分方程 $\frac{\mathrm{d}y}{\mathrm{d}t} = ky$。

□ 其他爲底數的指數函數用換底公式都可以改成以 e 爲底。

Population Growth, page 237

In the context of population growth, where $P(t)$ is the size of a population at time t, we can write

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP \quad \text{or} \quad \frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t} = k.$$

The quantity $\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t}$ is the growth rate divided by the population size; it is called the *relative growth rate* (相對成長率).

Instead of saying "the growth rate is proportional to population size" we could say "the relative growth rate is constant."

Example 3. The table gives the population of India, in millions, for the second half of the 20th century.

Year	1951	1961	1971	1981	1991	2001
Population	361	439	548	683	846	1029

(a) Use the exponential model and the census figure for 1951 and 1961 to predict the population in 2001. Compare with the actual population.

(b) Use the exponential model and the census figure for 1961 and 1981 to predict the population in 2001. Compare with the actual population. Then use this model to predict the population in the year 2010 and 2020.

Solution.

(a)

(b) $P(t) = P(0)\mathrm{e}^{kt} = 439\mathrm{e}^{kt}$, $P(20) = 439\mathrm{e}^{20k} = 683 \Rightarrow k = \frac{1}{20}\ln\frac{683}{439} \doteq 0.022099$.

$P(40) = 439\mathrm{e}^{40k} = 439\mathrm{e}^{0.88396} \doteq 1063$, $P(49) = 439\mathrm{e}^{49k} = 439\mathrm{e}^{1.08289} \doteq 1296$.

$P(59) = 439\mathrm{e}^{59k} = 439\mathrm{e}^{1.30389} \doteq 1617$.

Exercise.

(a) Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century.

(b) What is the relative growth rate?

(c) Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

Radioactive Decay, page 239

Radioactive substances decay by spontaneously emitting radiation.

If $m(t)$ is the mass remaining from an initial mass m_0 of the substance after time t, then the relative decay rate

$$-\frac{1}{m}\frac{\mathrm{d}m}{\mathrm{d}t}$$

has been found experimentally to be constant. It follows that

$$\frac{\mathrm{d}m}{\mathrm{d}t} = km,$$

where k is a negative constant. The solution is $m(t) = m_0 \mathrm{e}^{kt}$.

Physicist express the rate of decay in terms of *half-life* (半衰期), the time required for half of any given quantity to decay.

Example 4. Scientists can determine the age of ancient objects by the method of *radiocarbon dating* (放射性碳紀年). The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, $^{14}\mathrm{C}$, with a half-life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates $^{14}\mathrm{C}$ food chains. When a plant or animal dies, it stops replacing its carbon and the amount of $^{14}\mathrm{C}$ begins to decrease through radioactive decay. Therefore the level of radioactivity must also decay exponentially.

A parchment fragment was discovered that had about 74% as much $^{14}\mathrm{C}$ radioactivity as does plant material on the earth today. Estimate the age of the parchment.

Solution.

Exercise. Experiments show that if the chemical reaction

$$\mathrm{N_2O_5} \to 2\mathrm{NO_2} + \frac{1}{2}\mathrm{O_2}$$

takes place at 45°C, the rate of reaction of dinitrogen pentoxide is proportional to its concentration as follows:

$$-\frac{\mathrm{d}[\mathrm{N_2O_5}]}{\mathrm{d}t} = 0.0005[\mathrm{N_2O_5}]$$

(a) Find an expression for the concentration $[\mathrm{N_2O_5}]$ after t seconds if the initial concentration is C.

(b) How long will the reaction take to reduce the concentration of $\mathrm{N_2O_5}$ to 90% of its original value?

Newton's Law of Cooling, page 240

Newton's Law of Cooling (牛頓冷卻定律) states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provides that this difference is not too large.

If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings, then we can formulate Newton's Law of Cooling as a differential equation:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_s),$$

where k is a constant.

When we make the change of variable $y(t) = T(t) - T_s$. Since T_s is constant, we have $y'(t) = T'(t)$ and the equation becomes

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky.$$

Hence we can solve y first and then find T.

□ 牛頓冷卻定律也適用於物體增溫, 例如魚從冰箱拿至室溫解凍。

Example 5. In a murder investigation, the temperature of the corpse was 32.5°C at 1:30 PM and 30.3°C and hour later. Normal body temperature is 37.0°C and the temperature of the surroundings was 20.0°C. When did the murder take place?

Solution.

Exercise. A roast turkey is taken from an oven when its temperature has reached 85°C and is placed on a table in a room where the temperature is 22°C.

(a) If the temperature of the turkey is 65°C after half an hour, what is the temperature after 45 minutes?

(b) When will the turkey have cooled to 40°C?

Question 6. 兩杯熱咖啡, 一杯立馬加奶精, 然後放置五分鐘; 另一杯先放五分鐘後再加奶精, 哪一杯會比較熱?

Continuously Compounded Interest, page 241

If an amount A_0 is invested at an interest rate r, and interest is compounded n times a year, then in each compounding period the interest rate is $\frac{r}{n}$ and there are nt compounding periods in t years, so after t years the value of the investment is

$$A_0\left(1+\frac{r}{n}\right)^{nt}.$$

The interest paid increases as the number of compounding periods n increases. If we let $n \to \infty$, then we will be compounding the interest *continuously* (連續複利) and the value of the investment will be

$$A(t) = \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt} =$$

The above equation gives

$$\frac{\mathrm{d}A}{\mathrm{d}t} = rA(t),$$

which says that, with continuous compounding of interest, the rate of increase of an investment is proportional to its size.

Example 7.

(a) How long will it take an investment to double in value if the interest rate is 6% compounded continuously?

(b) What is the equivalent annual interest rate?

Solution.

Exercise.

(a) If \$3000 is invested at 5% interest, find the value of the investment at the end of 5 years if the interest is compounded (1) annually, (2) semiannually, (3) monthly, (4) weekly, (5) daily, and (6) continuously.

(b) If $A(t)$ is the amount of the investment at time t for the case of continuous compounding, write a differential equation and an initial condition satisfied by $A(t)$.