3.6 Derivatives of Logarithmic Function (page 218)

Another application of implicit differentiation is getting the derivatives of logarithmic functions.

Example 1 (page 218)**.** Compute $\frac{\mathrm{d}}{\mathrm{d}x}(\log_a x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)$.

Solution. Let $y = \log_a x$. Then $a^y = x$. Differentiating this equation implicit with respect to x, we get

In particular, we put $a = \mathrm{e}$ then $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \underline{\quad\quad}$.

Example 2 (page 220)**.** Find $f'(x)$ if $f(x) = \ln|x|$.

Solution. Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0, \end{cases} \quad \text{it follows that } f'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ & \text{if } x < 0. \end{cases}$$

Thus $f'(x) = \underline{\quad\quad}$ for all $x \neq 0$.

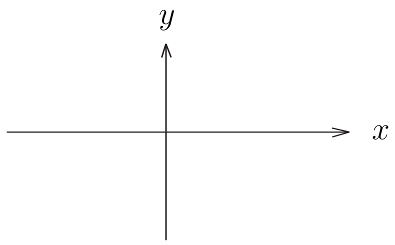

Figure 1: $f(x) = \ln|x|$.

☐ $f(x) = \ln|x|$ 的定義域是 $\underline{\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad}$.

Application: Logarithmic Differentiation (對數微分法)

The calculation of derivatives of complicated functions involving products, quotients, or powers can be simplified by the method of *logarithmic differentiation.*

Example 3 (page 220)**.** Differentiate $y = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$.

Solution.

The Power Rule (page 221)**.** *If* $n \in \mathbb{R}$ *and* $f(x) = x^n$*, then* $f'(x) = nx^{n-1}$.

Proof. Let $y = x^n$ and use logarithmic differentiation for $x \neq 0$:

If $x = 0$, by the definition of derivative, we have

$$f'(0) =$$

□

Example 4. Differentiate $y = x^x$. (The function is defined on $x > 0$)

Solution.

Solution 2.

□ 注意 x^n, a^x, x^x 變數的位置, 求導法則均不同。

The Number e as a Limit

Example 5 (page 189)**.** Show that $\mathrm{e} = \lim_{x \to 0}(1+x)^{\frac{1}{x}} = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$.

Solution. We have shown that if $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$. Thus $f'(1) = 1$. From the definition of a derivative as a limit and the continuity of the logarithmic function, we have

$$f'(1) = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x} =$$

$$=$$

Hence we have $\lim_{x \to 0}(1+x)^{\frac{1}{x}} = \mathrm{e}$. If we put $n = \frac{1}{x}$, then $n \to \infty$ as $x \to 0^+$, then we get an alternative expression for e:

$$\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = \mathrm{e}.$$

Example 6 (page 189)**.** Show that $\lim_{n \to \infty}\left(1 + \frac{x}{n}\right)^n = \mathrm{e}^x$ for any $x > 0$.

Solution.