3.5 Implicit Differentiation (page 208)

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable $y = f(x)$. However, there are a lot of functions are defined implicitly by a relation x and y and we formally write it as $F(x, y) = 0$. For example, $F(x, y) = x^2 + y^2 - 4 = 0$, $F(x, y) = x^3 + y^3 - 6xy = 0$.

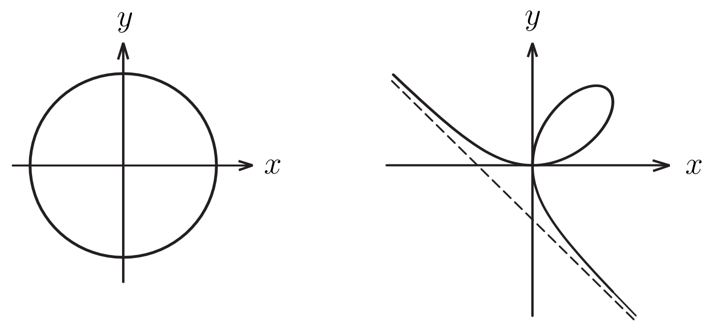

Figure 1: (a) A circle $x^2 + y^2 = 4$. (b) The folium of Descartes $x^3 + y^3 - 6xy = 0$.

Most of time, implicit functions are not "functions" (see the definition of a function in Section 1.1), but they are locally be expressed as functions.

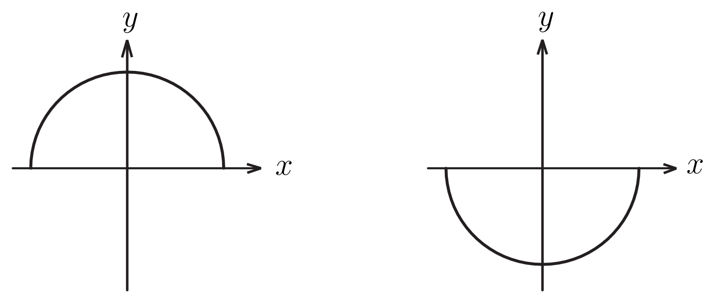

Figure 2: A circle $x^2 + y^2 = 4$.

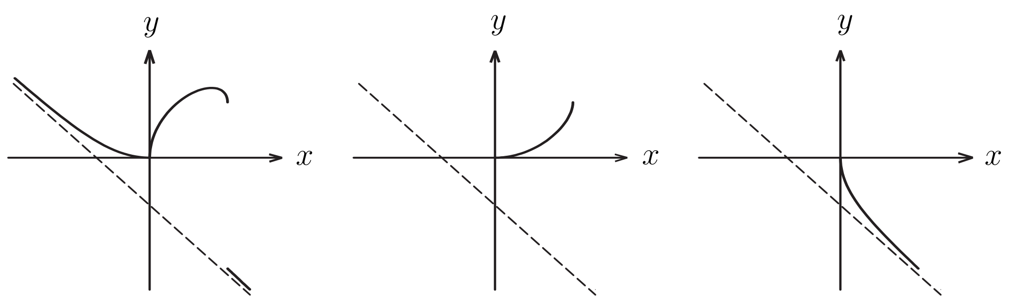

Figure 3: The folium of Descartes $x^3 + y^3 - 6xy = 0$.

Furthermore, it's not easy to solve implicit functions $F(x, y) = 0$ to explicit ones $y = f(x)$. Fortunately, we can compute the derivative of implicit functions without solving implicit functions to explicit ones by ______________.

Example 1. If $x^2 + y^2 = r^2$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution.

Solution 2.

☐ 將隱函數 $F(x, y) = 0$ 視爲 $F(x, y(x)) = 0$。

Example 2.

(a) Find y' if $x^3 + y^3 = 6xy$.

(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

(c) At what point in the first quadrant is the tangent line horizontal?

Solution.

If we solve the equation $x^3 + y^3 = 6xy$ for y in terms of x, we get three functions determined by the equation:

$$y = f(x) = \sqrt[3]{-\frac{1}{2}x^3 + \sqrt{\frac{1}{4}x^6 - 8x^3}} + \sqrt[3]{-\frac{1}{2}x^3 - \sqrt{\frac{1}{4}x^6 - 8x^3}}$$

and

$$y = \frac{1}{2}\left(-f(x) \pm \sqrt{-3}\left(\sqrt[3]{-\frac{1}{2}x^3 + \sqrt{\frac{1}{4}x^6 - 8x^3}} - \sqrt[3]{-\frac{1}{2}x^3 - \sqrt{\frac{1}{4}x^6 - 8x^3}}\right)\right).$$

It is very complicated to get the derivative by these formulae.

Implicit differentiation works for a lot of equations such as $y^5 + 3x^2y^2 + 5x^4 = 12$ for which it is *impossible* to find an expression for y in terms of x.

An application of implicit differentiation is derivatives of inverse functions.

Derivatives of Inverse Trigonometric Functions (page 214)**.**

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}x = \frac{1}{1+x^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cot^{-1}x = -\frac{1}{1+x^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\csc^{-1}x = -\frac{1}{x\sqrt{x^2-1}}.$$

Proof. Let $y = y(x) = \sin^{-1}x$, then $\sin y = x \Rightarrow \cos y \frac{\mathrm{d}y}{\mathrm{d}x} = 1$. So

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

□

□ $\sin^{-1}x, \cos^{-1}x, \tan^{-1}x$ 的導函數要熟記; 六個反三角函數的導函數也要會推導。

Example 3 (Derivatives of inverse functions)**.** Suppose f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Show that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provide that the denominator is not 0.

Solution.

Example 4. Find the tangent line of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ at (x_0, y_0) and the length between x-intercept and y-intercept.

Solution.

Example 5. Suppose that $f(x) \in C^2(\mathbb{R})$ and $f(x)$ satisfies $x^2+xf(x)+(f(x))^2 = k$, where k is a constant, and $f'(a) = f''(a) = 1$. Find a and k.

Solution.

Example 6. Two curves are *orthogonal* (正交) if their tangent lines are perpendicular at each point of intersection. Show that the given families of curves are *orthogonal trajectories* (正交軌線) of each other; that is, every curve is one family is orthogonal to every curve in the other family.

(a) $x^2 + y^2 = r^2$, $ax + by = 0$.

(b) $x^2 + y^2 = ax$, $x^2 + y^2 = by$.

Solution.

(a)

(b) First, $x^2 + y^2 = ax \Rightarrow 2x + 2yy' = a \Rightarrow y' = \frac{a-2x}{2y}$ if $y \neq 0$. Next, $x^2 + y^2 = by \Rightarrow 2x + 2yy' = by' \Rightarrow (b-2y)y' = 2x \Rightarrow y' = \frac{2x}{b-2y}$ if $y \neq \frac{b}{2}$. So if $y \neq 0$ and $y \neq \frac{b}{2}$, we have

$$m_1 \cdot m_2 = \frac{a-2x}{2y} \cdot \frac{2x}{b-2y} = \frac{2ax-4x^2}{2by-4y^2} = \frac{2x^2+2y^2-4x^2}{2x^2+2y^2-4y^2} = \frac{2y^2-2x^2}{2x^2-2y^2} = -1.$$

If $y = 0$, then $x^2 - ax = x(x-a) = 0 \Rightarrow x = 0$ or $x = a$, so $x^2 + y^2 = ax$ has vertical tangent line at $x = 0$ or $x = a$. If $(x, y) = (0, 0)$, $m_2 = 0$. If $(x, y) = (a, 0), a \neq 0$, no curves in the family $x^2 + y^2 = by$ passes through $(a, 0)$. If $y = \frac{b}{2}$, then $x = \pm\frac{b}{2}$, so $x^2 + y^2 = by$ has vertical tangent line at $(\frac{b}{2}, \pm\frac{b}{2})$. At $(\frac{b}{2}, \pm\frac{b}{2})$, we get $a = b$, and $m_1 = a - 2x = b - 2\frac{b}{2} = 0$, so $x^2 + y^2 = ax$ has horizontal tangent line at $(\frac{b}{2}, \pm\frac{b}{2})$.