3.4 The Chain Rule (page 197)

Theorem 1 (The Chain Rule (鏈鎖律), page 198)**.** *If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product*

$$F'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}.$$

Theorem 2 (The power rule combined with the chain rule, page 200)**.** *If n is any real number and $g(x)$ is differentiable, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}(g(x))^n = n(g(x))^{n-1} \cdot g'(x).$$

Proof. The relation is $x \xrightarrow{g} u = g(x) \xrightarrow{f} y = u^n$. By the Chain Rule, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left.\frac{\mathrm{d}y}{\mathrm{d}u}\right|_{u=g(x)} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = \left. nu^{n-1}\right|_{u=g(x)} \cdot g'(x) = n(g(x))^{n-1} \cdot g'(x).$$

□

- □ 合成函數的求導法則。
- □ 熟悉合成函數的先後順序。
- □ 洋蔥: 如果你願意一層一層一層的剝開我的心 ......

Example 3.

(a) $(\sin(ax))' =$

(b) $(\sin(x^2))' =$

(c) $(\sin^2 x)' =$

(d) $(\mathrm{e}^{2x})' =$

(e) $(\mathrm{e}^{x^2})' =$

Example 4 (page 202)**.** Show that $\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x \ln a$.

Solution.

Example 5. Let $f(x) = \frac{x}{\sqrt[3]{3x+5}}$. Find $f'(1)$.

Solution.

☐ 熟悉除法法則、鏈鎖率、代值。

Example 6. Let $y = x^{a^b} + a^{x^b} + a^{b^x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. (96,98 微甲一組)

Solution.

☐ 想清楚這三個函數的意義與合成函數的先後順序; 清楚何時使用多項式或指數微分。

Example 7. Let $f(x), g(x) \in C^1(\mathbb{R})$ and $f(1) = 1, f'(1) = 0, g(0) = 0, g'(0) = 1$. Find the limit

$$\lim_{x \to \frac{\pi}{2}} \frac{f(\sin x) g(\cos x)}{x^2 - \frac{\pi}{2} x}.$$

(91 微甲一組)

Solution.

☐ 將極限問題與某個函數的導數的定義連接起來。

Example 8. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

(a) Find $f'(0)$.

(b) When $x \neq 0$, find $f'(x)$.

(c) Does $f''(0)$ exist? If it exists, please find its value. If not, give reason to support your argument. (89 微甲一組)

Solution.

☐ (a) 遇到分段定義的函數, 其導數利用「定義」處理。

☐ (b) 導函數的四則運算與鏈鎖率是建立在函數都「很好」才能使用。

☐ (c) 高階導數, 遇到分段定義的函數, 仍然是由「定義」出發。

☐ 比較 Section 2.7 的 **Example 6**。

To prove the chain rule, one idea is the following:

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\ &= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(g(x)) \cdot g'(x). \end{aligned}$$

This argument looks great, but it is *not* correct. What is the problem? How do we overcome the problem?

Appendix

Proof of the Chain Rule. The function $g(x)$ is differentiable at x. This means $g'(x)$ exists and

$$\frac{g(x+h)-g(x)}{h} - g'(x) \to 0 \quad \text{as} \quad h \to 0.$$

Define a new variable v by

$$v = \frac{g(x+h)-g(x)}{h} - g'(x) \Rightarrow g(x+h) = g(x) + (g'(x)+v)h. \tag{1}$$

Notice that v depends on the number h and that $v \to 0$ as $h \to 0$. Similarly, because the function f is differentiable at the point $y = g(x)$, we have

$$\frac{f(y+k)-f(y)}{k} - f'(y) \to 0 \quad \text{as} \quad k \to 0.$$

Define another variable w by

$$w = \frac{f(y+k)-g(y)}{k} - f'(y) \Rightarrow f(y+k) = f(y) + (f'(y)+w)k. \tag{2}$$

Notice that w depends on the number k and that $w \to 0$ as $k \to 0$.

From (1), we get

$$f(g(x+h)) = f(g(x) + (g'(x)+v)h).$$

Use (2) applied to the right-hand-side with $k = (g'(x)+v)h$ and $y = g(x)$ to get.

$$f(g(x) + (g'(x)+v)h) = f(g(x)) + (f'(g(x))+w)(g'(x)+v)h.$$

Note that $k \to 0$ as $h \to 0$, and so $w \to 0$ as $h \to 0$. So

$$\begin{aligned}\frac{f(g(x+h))-f(g(x))}{h} &= \frac{f(g(x)) + (f'(g(x))+w)(g'(x)+v)h - f(g(x))}{h} \\ &= \frac{(f'(g(x))+w)(g'(x)+v)h}{h} = (f'(g(x))+w)(g'(x)+v).\end{aligned}$$

Hence

$$\begin{aligned}&\lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h} = \lim_{h\to 0}(f'(g(x))+w)(g'(x)+v) \\ &= \left(\lim_{h\to 0} f'(g(x)) + \lim_{h\to 0} w\right)\left(\lim_{h\to 0} g'(x) + \lim_{h\to 0} v\right) = f'(g(x))g'(x).\end{aligned}$$

since $v \to 0$ as $h \to 0$ and $w \to 0$ as $h \to 0$. □