3.3 Derivatives of Trigonometric Functions (page 190)

<u>Goal</u>: Find the derivative of six trigonometric functions.

Example 1 (page 192–193)**.** Calculate the derivative of $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$.

Solution. We compute

$$\begin{aligned} f'(\theta) &= \lim_{h\to 0} \frac{f(\theta+h)-f(\theta)}{h} = \lim_{h\to 0} \frac{\sin(\theta+h)-\sin\theta}{h} = \lim_{h\to 0} \frac{2\cos\left(\theta+\frac{h}{2}\right)\sin\frac{h}{2}}{h} \\ &= \lim_{h\to 0} \cos\left(\theta+\frac{h}{2}\right) \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} = \cos\theta \\ g'(\theta) &= \lim_{h\to 0} \frac{g(\theta+h)-g(\theta)}{h} = \lim_{h\to 0} \frac{\cos(\theta+h)-\cos\theta}{h} = \lim_{h\to 0} \frac{-2\sin\left(\theta+\frac{h}{2}\right)\sin\frac{h}{2}}{h} \\ &= \lim_{h\to 0} -\sin\left(\theta+\frac{h}{2}\right) \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} = -\sin\theta. \end{aligned}$$

Example 2 (page 193)**.** Calculate the derivative of $\tan\theta, \cot\theta, \sec\theta$, and $\csc\theta$.

Solution. Using the quotient rule, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta}\tan\theta &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\sin\theta}{\cos\theta}\right) = \frac{\cos\theta(\sin\theta)' - \sin\theta(\cos\theta)'}{\cos^2\theta} = \frac{\cos\theta\cos\theta - \sin\theta(-\sin\theta)}{\cos^2\theta} \\ &= \frac{1}{\cos^2\theta} = \sec^2\theta \\ \frac{\mathrm{d}}{\mathrm{d}\theta}\cot\theta &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{\tan\theta}\right) = \frac{(\tan\theta)(1)' - 1(\tan\theta)'}{\tan^2\theta} = -\frac{\sec^2\theta}{\tan^2\theta} = -\csc^2\theta \\ \frac{\mathrm{d}}{\mathrm{d}\theta}\sec\theta &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{\cos\theta}\right) = \frac{(\cos\theta)(1)' - 1(\cos\theta)'}{\cos^2\theta} = \frac{\sin\theta}{\cos^2\theta} = \sec\theta\tan\theta \\ \frac{\mathrm{d}}{\mathrm{d}\theta}\csc\theta &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{\sin\theta}\right) = \frac{(\sin\theta)(1)' - 1(\sin\theta)'}{\sin^2\theta} = -\frac{\cos\theta}{\sin^2\theta} = -\csc\theta\cot\theta. \end{aligned}$$

☐ $\tan\theta\cot\theta = 1 \Rightarrow \sec^2\theta\cot\theta + \tan\theta(\cot\theta)' = 0 \Rightarrow (\cot\theta)' = -\frac{1}{\sin^2\theta} = -\csc^2\theta$.

☐ 務必熟記六個三角函數的導函數。

Example 3. Calculate the derivative of $\sin^2\theta$ and $\cos^2\theta$.

Solution. We compute

$$(\sin^2\theta)' = (\sin\theta\sin\theta)' = \cos\theta\sin\theta + \sin\theta\cos\theta = 2\sin\theta\cos\theta.$$

$$(\cos^2\theta)' = (\cos\theta\cos\theta)' = -\sin\theta\cos\theta - \cos\theta\sin\theta = -2\sin\theta\cos\theta.$$

☐ $\sin^2\theta + \cos^2\theta = 1 \Rightarrow (\sin^2\theta + \cos^2\theta)' = 0$.

Example 4. Compute $\frac{\mathrm{d}^n}{\mathrm{d}\theta^n}\sin\theta$ and $\frac{\mathrm{d}^n}{\mathrm{d}\theta^n}\cos\theta$.

Solution. Direct computation gives

$$\begin{aligned}
f(\theta) &= \sin\theta = f^{(4)}(\theta) = f^{(4k)}(\theta) & g(\theta) &= \cos\theta = g^{(4)}(\theta) = g^{(4k)}(\theta)\\
f'(\theta) &= \cos\theta = f^{(5)}(\theta) = f^{(4k+1)}(\theta) & g'(\theta) &= -\sin\theta = g^{(5)}(\theta) = g^{(4k+1)}(\theta)\\
f''(\theta) &= -\sin\theta = f^{(6)}(\theta) = f^{(4k+2)}(\theta) & g''(\theta) &= -\cos\theta = g^{(6)}(\theta) = g^{(4k+2)}(\theta)\\
f'''(\theta) &= -\cos\theta = f^{(7)}(\theta) = f^{(4k+3)}(\theta) & g'''(\theta) &= \sin\theta = g^{(7)}(\theta) = g^{(4k+3)}(\theta).
\end{aligned}$$

Remark that we have another type of formula:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta = \sin\left(\theta+\frac{\pi}{2}\right) &\Rightarrow \frac{\mathrm{d}^n}{\mathrm{d}\theta^n}\sin\theta = \sin\left(\theta+\frac{n\pi}{2}\right)\\
\frac{\mathrm{d}}{\mathrm{d}\theta}\cos\theta = \cos\left(\theta+\frac{\pi}{2}\right) &\Rightarrow \frac{\mathrm{d}^n}{\mathrm{d}\theta^n}\cos\theta = \cos\left(\theta+\frac{n\pi}{2}\right).
\end{aligned}$$

以下求導法則務必熟記

- $(c)' = 0.$ — Derivative of a constant function
- $(x^n)' = nx^{n-1}.$ — The powe rule
- $(cf(x))' = cf'(x).$ — The constant multiple rule
- $(f(x)+g(x))' = f'(x)+g'(x).$ — The sum rule
- $(f(x)-g(x))' = f'(x)-g'(x).$ — The difference rule
- $(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$ — The produce rule
- $\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}.$ — The quotient rule
- ⋆ $(f(g(x)))' = f'(g(x))\cdot g'(x).$ — The chain rule

三角函數的導函數, 務必熟記

$(\sin\theta)' = \cos\theta$	$(\tan\theta)' = \sec^2\theta$	$(\sec\theta)' = \sec\theta\tan\theta$
$(\cos\theta)' = -\sin\theta$	$(\cot\theta)' = -\csc^2\theta$	$(\csc\theta)' = -\csc\theta\cot\theta$

e 的定義與 e^x 的導函數

$$\lim_{h\to 0}\frac{\mathrm{e}^h-1}{h} = 1, \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x) = \mathrm{e}^x.$$