3.2 The Product and Quotient Rules (page 183)

Property 1 (The product rule, page 184)**.** *If f and g are both differentiable, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) = g(x)\frac{\mathrm{d}}{\mathrm{d}x}f(x) + f(x)\frac{\mathrm{d}}{\mathrm{d}x}g(x).$$

Proof. Let $F(x) = f(x)g(x)$, then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)g(x+h)+f(x)\left(\frac{g(x+h)-g(x)}{h}\right) \\
&= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)\lim_{h\to 0}g(x+h)+f(x)\lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right) \\
&= f'(x)g(x)+f(x)g'(x).
\end{aligned}$$

□

□ 注意乘法法則 $(f(x)g(x))' \neq f'(x)g'(x)$。

□ 推廣: $(f(x)g(x)h(x))' = \underline{f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)}$.

□ 萊布尼茲法則 (Leibniz Rule): $(fg)^{(n)}(x) = \sum_{k=0}^{n} C_k^n f^{(n-k)}(x)g^{(k)}(x)$.

Property 2 (The quotient rule, page 186)**.** *If f and g are both differentiable, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{\mathrm{d}}{\mathrm{d}x}f(x) - f(x)\frac{\mathrm{d}}{\mathrm{d}x}g(x)}{(g(x))^2}.$$

Proof. Let $F(x) = \frac{f(x)}{g(x)}$. Then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{f(x+h)g(x)-g(x+h)f(x)}{hg(x+h)g(x)} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-g(x+h)f(x)}{hg(x+h)g(x)} \\
&= \lim_{h\to 0}\frac{(f(x+h)-f(x))g(x)-f(x)(g(x+h)-g(x))}{hg(x+h)g(x)} \\
&= \lim_{h\to 0}\frac{\frac{f(x+h)-f(x)}{h}g(x)-f(x)\frac{g(x+h)-g(x)}{h}}{g(x+h)g(x)} \\
&= \frac{\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}g(x)-f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}}{\lim_{h\to 0}g(x+h)g(x)} = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}.
\end{aligned}$$

□

□ 分子的兩項係數, 一正一負, 如何確定何者爲正何者爲負?

Example 3. Compute $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{f(x)}{g(x)}\right)$.

Solution. We compute

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{f(x)}{g(x)}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{gf'-fg'}{g^2}\right)\\ &= \frac{g^2(gf'-fg')'-(gf'-fg')(g\cdot g)'}{g^4}\\ &= \frac{g^2(g'f'+gf''-f'g'-fg'')-(gf'-fg')(2g'g)}{g^4}\\ &= \frac{g(gf''-fg'')-2g'(gf'-fg')}{g^3}\end{aligned}$$

□ 函數 $\frac{f(x)}{g(x)}$ 微分 n 次, 分母一定可以化簡成 $(g(x))^{n+1}$。

Example 4. The curve $y=\frac{1}{1+x^2}$ is called a *witch of Maria Agnesi* (箕舌線). Find an equation of the tangent line to this curve at the point $(-1,\frac{1}{2})$.

Solution. Since

$$y'(x)=\frac{(1+x^2)(1)'-1(1+x^2)'}{(1+x^2)^2}=\frac{-2x}{(1+x^2)^2},\quad y'(-1)=\frac{(-2)\cdot(-1)}{(1+(-1)^2)^2}=\frac{1}{2}.$$

we know the tangent line equation is $y-\frac{1}{2}=\frac{1}{2}(x+1)$.