Chapter 3 Differentiation Rules

3.1 Derivatives of Polynomials and Exponential Functions (page 172)

Property 1 (Derivative of a constant function, page 172)**.**

$$\frac{\mathrm{d}}{\mathrm{d}x}(c) = 0.$$

Proof. Let $f(x) = c$ the constant function, then from the definition of a derivative, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$

□

Property 2 (The power rule, page 173)**.** *If n is any real number, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}.$$

Proof. Let $f(x) = x^n$. Here we check the case $n \in \mathbb{Z}$ and show the general case in Section 3.6. First, for $n \in \mathbb{N}$, by the Binomial Theorem, we compute

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{\sum_{k=0}^{n} C_k^n x^{(n-k)} h^k - x^n}{h} \\ &= \lim_{h\to 0} \sum_{k=1}^{n} C_k^n x^{(n-k)} h^{k-1} = \sum_{k=1}^{n} \left(\lim_{h\to 0} C_k^n x^{(n-k)} h^{k-1} \right) = C_1^n x^{n-1} = nx^{n-1}. \end{aligned}$$

Next, we check the case negative integer $-n, n \in \mathbb{N}$. That is, let $f(x) = x^{-n}$, and we will prove $f'(x) = -nx^{-n-1}$:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1}{h} \left(\frac{1}{(x+h)^n} - \frac{1}{x^n} \right) \\ &= \lim_{h\to 0} -\frac{\sum\limits_{k=0}^{n} C_k^n x^{(n-k)} h^k - x^n}{h(x+h)^n x^n} = \lim_{h\to 0} -\sum_{k=1}^{n} \frac{C_k^n x^{(n-k)} h^{k-1}}{(x+h)^n x^n} \\ &= -\sum_{k=1}^{n} \left(\lim_{h\to 0} \frac{C_k^n x^{(n-k)} h^{k-1}}{(x+h)^n x^n} \right) = -C_1^n x^{-n-1} = -nx^{-n-1}. \end{aligned}$$

□

☐ n 次方差公式: $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$。

☐ 活用公式: $a - b = (a^{\frac{1}{n}} - b^{\frac{1}{n}})(a^{\frac{n-1}{n}} + a^{\frac{n-2}{n}} b^{\frac{1}{n}} + \cdots + a^{\frac{1}{n}} b^{\frac{n-2}{n}} + b^{\frac{n-1}{n}})$。

Property 3 (The constant multiple rule, page 175)**.** *If c is a constant and f is a differential function, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}(cf(x)) = c\frac{\mathrm{d}}{\mathrm{d}x}f(x).$$

Proof. Let $g(x) = cf(x)$. Then

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = \lim_{h\to 0} c\left(\frac{f(x+h)-f(x)}{h}\right) \\ &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = cf'(x). \end{aligned}$$

□

Property 4 (The sum and difference rule, page 176)**.** *If f and g are both differentiable, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)\pm g(x)) = \frac{\mathrm{d}}{\mathrm{d}x}f(x) \pm \frac{\mathrm{d}}{\mathrm{d}x}g(x).$$

Proof. Let $F(x) = f(x) \pm g(x)$. Then

$$\begin{aligned} F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{(f(x+h)\pm g(x+h))-(f(x)\pm g(x))}{h} \\ &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h} \pm \frac{g(x+h)-g(x)}{h}\right) \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \pm \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f'(x) \pm g'(x). \end{aligned}$$

□

Example 5. Compute the derivative of the exponential function $f(x) = a^x$.

Solution.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = \lim_{h\to 0}\frac{a^x a^h - a^x}{h} \\ &= \lim_{h\to 0}\frac{a^x(a^h-1)}{h} = \left(\lim_{h\to 0}\frac{a^h-1}{h}\right)a^x = f'(0)f(x). \end{aligned}$$

Definition 6 (the number e, page 180)**.** e is the number such that

$$\lim_{h\to 0}\frac{\mathrm{e}^h-1}{h} = 1.$$

□ $\mathrm{e} \approx 2.71828\ldots$ (回想 section 1.5 的介紹, 到 section 3.6 會證明 $\lim_{n\to\infty}(1+\frac{1}{n})^n = \mathrm{e}$)。

Property 7 (Derivative of the natural exponential function, page 178)**.**

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x) = \mathrm{e}^x.$$

Proof. Let $f(x) = \mathrm{e}^x$, then

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\mathrm{e}^x(\mathrm{e}^h-1)}{h} = \left(\lim_{h\to 0}\frac{\mathrm{e}^h-1}{h}\right)\mathrm{e}^x = \mathrm{e}^x.$$

□