2.8 The Derivative as a Function (page 152)

Definition 1 (page 152)**.** The *derivative of* $f(x)$ ($f(x)$ 的導函數) is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

☐ 導函數 $f'(x)$ 的定義域是 $\{x \in \mathbb{R} | f'(x) \text{ exists}\}$。

☐ 左導函數記爲 $f'_{-}(x) = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$; 右導函數記爲 $f'_{+}(x) = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$。

☐ 導函數存在等價於左導函數與右導函數皆存在且相等。

Example 2. Let $f(x) = x^3$. Find $f'(x)$.

Solution. By definition,

Other Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = Df(x) = D_x f(x).$$

The symbols D and $\frac{\mathrm{d}}{\mathrm{d}x}$ are called *differentiation operators* (微分算子) because thy indicate the operation of *differentiation*.

We use the notation

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a} \quad \text{or} \quad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{x=a}$$

to indicate the value of a variable $\frac{\mathrm{d}y}{\mathrm{d}x}$ at a specific number a, which is a synonym for $f'(a)$.

Definition 3 (page 155)**.** A function f is *differentiable at* a (在 $x = a$ 處可微分) if $f'(a)$ exists. It is *differentiable on an open interval* (a, b) (or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$) if it is differentiable at every number in the interval.

☐ 若函數只定義於 $[a, b]$, 則端點的導數就只要看 $\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$ 及 $\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h}$。

Theorem 4 (page 157)**.** *If f is differentiable at a, then f is continuous at a.*

Proof of **Theorem 4** is in the Appendix.

☐ Theorem 4 的逆敘述不對, 例如: $f(x) = |x|$。

☐ 數學上有一種函數是處處連續, 但處處不可微分。

How Can a Function Fail to Be Differentiable?

(1) corner or kink: the graph of f has no tangent at this point and f is not differentiable there.

(2) discontinuity: f is not continuous at a, then f is not differentiable at a.

(3) vertical tangent line: f is continuous at a and $\lim\limits_{x\to a} |f'(x)| = \infty$.

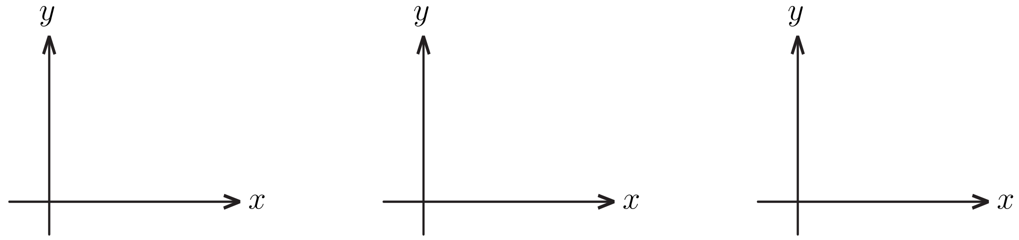

Figure 1: Three ways for $f(x)$ not to be differentiable at $x = a$.

Higher Derivatives

If f is differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the *second derivative* of f (二次導數). We write the second derivative of $y = f(x)$ as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}.$$

☐ *acceleration* (加速度): 速度函數對時間的瞬間變化率。

The *third derivative* f''' (三次導數) is the derivative of the second derivative: $f''' = (f'')'$. If $y = f(x)$, then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = \frac{\mathrm{d}^3y}{\mathrm{d}x^3}.$$

In general, the *n-th derivative* ($n \geq 4$) of f is denoted by $f^{(n)}$. If $y = f(x)$, we write

$$y^n = f^{(n)}(x) = \frac{\mathrm{d}^ny}{\mathrm{d}x^n}.$$

☐ *jerk* (急動度、加加速度): 加速度的變化率。

Example 5. Suppose

$$f(x) = \begin{cases} \frac{1-\cos x}{\sin x} & x > 0 \\ ax + b & x \leq 0. \end{cases}$$

Find a and b such that f is continuous and differentiable at $x = 0$.

Solution.

☐ 想清楚函數在一個點「連續」、「可微分」的意義 (數學定義)。

Example 6. Let $f(x) = x|x|$. Find $f'(x)$ and $f''(x)$.

Solution.

☐ 遇到分段定義的函數 (例如這個例子中的 $x = 0$ 處) 必須「用定義」小心處理。

☐ 割線斜率的極限 $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ 和切線斜率的極限 $\lim_{h\to 0} f'(x+h)$ 是兩個不同的概念。

☐ 記符號 $C^k(\mathbb{R})$ 爲所有 k 次求導後仍爲連續的函數所成的集合。

☐ 若 $f(x) \in C^1(\mathbb{R})$, 則有 $\lim_{x\to a} f'(x) = f'(\lim_{x\to a} x) = f'(a)$。

☐ 結論: $f(x) = x|x| \in C^1(\mathbb{R})$。

Appendix

Proof of Theorem 4. The goal is to show that $\lim_{x\to a} f(x) = f(a)$.
For $x \neq a$, we have

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a} \cdot (x - a),$$

so

$$\begin{aligned}
\lim_{x\to a}(f(x) - f(a)) &= \lim_{x\to a}\left(\frac{f(x) - f(a)}{x - a} \cdot (x - a)\right) = \lim_{x\to a}\frac{f(x) - f(a)}{x - a} \cdot \lim_{x\to a}(x - a) \\
&= f'(a) \cdot 0 = 0.
\end{aligned}$$

Hence

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a}(f(x) - f(a) + f(a)) = \lim_{x\to a}(f(x) - f(a)) + \lim_{x\to a} f(a) \\
&= 0 + f(a) = f(a).
\end{aligned}$$

□