2.7 Derivatives and Rates of Change (page 140)

Definition 1 (page 141)**.** The *tangent line* (切線) to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope (斜率)

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that this limit exists.

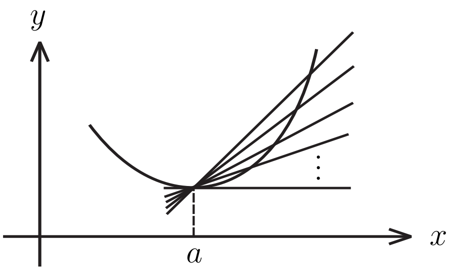

Figure 1: Tangent line is the limiting position of the secant line (割線).

☐ 切線斜率的定義是 ______________________。

Example 2. Find an equation of the tangent line to the hyperbola $y = \frac{1}{x}$ at the point $(1, 1)$.

Solution. Let $f(x) = \frac{1}{x}$. Then the slope of the tangent at $(1, 1)$ is

Therefore, an equation of the tangent at the point $(1, 1)$ is ________________.

Definition 3 (page 143)**.** If $f(x)$ is the position function, then the *average velocity* (平均速度) is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h},$$

and the *velocity* (or *instantaneous velocity*, 瞬時速度) $v(a)$ at time $t = a$ be the limit of these average velocities:

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

☐ The *speed* (速率) of the particle is the absolute value of the velocity $|v(a)|$.

Example 4. Suppose that a ball is dropped from the upper observation deck of Taipei 101, 508m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

Solution. Using the equation of motion $s = f(t) = 4.9t^2$, we have

(a) The velocity after 5 is __________.
(b) First we solve $4.9t_1^2 = 508$. This gives $t_1 = \sqrt{\frac{508}{4.9}}$. The velocity of the ball as it hits the ground is

Definition 5 (page 144)**.** The *derivative of a function* $f(x)$ *at a number* $x = a$ (函數 $f(x)$ 在 $x = a$ 的導數), denoted by $f'(a)$, is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

If we use the point-slope form (點斜式) of the equation of a line, we can write an equation of the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$:

$$y - f(a) = f'(a)(x - a).$$

☐ 接下來的章節將討論如何計算各種函數的導數、導函數及其性質。

Example 6. Consider

$$f(x) = \begin{cases} x^\alpha \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

where α is a natural number. Determine whether $f'(0)$ exists.

Solution. By the definition of derivative, we have

☐ 這個例題很重要, 之後還有延伸的問題, 務必弄清楚。

Rates of Change

Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the *increment* (增加量) of x) is $\Delta x = x_2 - x_1$, and the corresponding change in y is $\Delta y = f(x_2) - f(x_1)$. The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the *average of the change of y with respect to x* (平均變化率) over the interval $[x_1, x_2]$. We say

$$\text{instantaneous rate of change } = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$ (瞬間變化率).

Examples of rates of change:

(1) Velocity of an object: the rate of change of displacement with respect to time.

(2) Marginal cost (邊際成本): the rate of change of production cost with respect to the number of items produced.

(3) Interest (in economics): the rate of change of the debt with respect to time.

(4) Power (in physics, 功率): the rate of change of work with respect to time.

(5) Rate of reaction (in chemistry): the rate of change in the concentration (濃度) of a reactant with respect to time.

(6) Rate of change of the population of a colony of bacteria with respect to time. (biology)