2.6 Limits at Infinity: Horizontal Asymptotes (page 126)

Definition 1 (page 127–128)**.** Let f be a function defined on real numbers.

(a) $\lim_{x\to\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large (無窮遠處之極限).

(b) $\lim_{x\to-\infty} f(x) = L$ means that the value of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative. (負無窮遠處之極限)

☐ 極限 $\lim_{x\to\infty} f(x) = L$ 的另一種表達法爲 "$f(x) \to L$ as $x \to \infty$"。

☐ 極限 $\lim_{x\to-\infty} f(x) = L$ 的另一種表達法爲 "$f(x) \to L$ as $x \to -\infty$"。

Definition 2 (page 128)**.** The line $y = L$ is called a *horizontal asymptote* (水平漸近線) of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

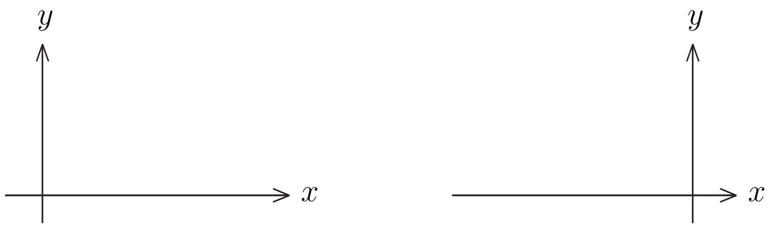

Figure 1: Limit $\lim_{x\to\infty} f(x) = L$, $\lim_{x\to-\infty} f(x) = L$, and horizontal asymptotes.

☐ 若一對一函數有垂直漸近線, 則其反函數有水平漸近線。例如: ______________。

Theorem 3 (page 129)**.**

(a) *If $r > 0$ is a rational number, then* $\lim_{x\to\infty} \frac{1}{x^r} = 0$.

(b) *If $r > 0$ is a rational number such that x^r is defined for all x, then* $\lim_{x\to-\infty} \frac{1}{x^r} = 0$.

Example 4. Evaluate $\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1}$.

Solution.

☐ 設 $P(x), Q(x)$ 為兩多項式, 其領導係數分別為 a_n 與 b_m, 則

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \begin{cases} & \text{if} \\ & \text{if} \\ & \text{if} \end{cases}$$

Example 5. Evaluate $\lim\limits_{x\to\infty}(2^x + 3^x + 5^x)^{\frac{1}{x}}$.

Solution.

☐ 若有正數 $a_1, a_2, \ldots a_n$, 則 $\lim\limits_{x\to\infty}((a_1)^x + (a_2)^x + \cdots + (a_n)^x)^{\frac{1}{x}} =$____________.

Example 6. Find the following limit:

(a) $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$ (b) $\lim\limits_{x\to\infty} \dfrac{\sin x}{x}$ (c) $\lim\limits_{x\to 0} x \sin \dfrac{1}{x}$ (d) $\lim\limits_{x\to\infty} x \sin \dfrac{1}{x}$.

Solution.

(a) $\lim\limits_{x\to 0} \frac{\sin x}{x} =$ ___. (See section 2.3 **Example 9.**)

(b)

(c)

(d)

☐ 這個例題的心得是: ____________________________________

Example 7. Evaluate the limit $\lim\limits_{x\to\infty} \frac{3x^2+5}{5x+3} \sin \frac{2}{x}$.

Solution.

Example 8. Let $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \mathrm{e}$. Find $\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x$. (97 微甲一組)

Solution.

Example 9. Find the limit $\lim_{x\to 0^+}(\cos\sqrt{x})^{\frac{1}{x}}$.

Solution.

☐ 學會變數變換與湊變數的能力。

Example 10. Let $f(x)=\sqrt{x^2+x}$. Compute the following limits:

(a) $A=\lim_{x\to\infty}\frac{f(x)}{x}$.

(b) $\lim_{x\to\infty}(f(x)-Ax)$.

Solution.

☐ 函數在無窮遠的行爲, 有一類是存在「斜漸近線」。例如標準型的雙曲線 $x^2-y^2=1$。

Example 11. Suppose α, β are two constants and $\lim_{x\to-\infty}(\sqrt{x^2+3x+2}-\alpha x-\beta)=0$. Find α and β. (91 微甲一組)

Solution.

☐ 計算 $x \to -\infty$ 的極限時要特別小心 x 是小於零的數。

☐ 害怕直接處理 $x \to -\infty$ 極限會出錯的話, 可以考慮用變數變換:

Infinite Limits at Infinity, Precise Definition

Definition 12 (page 134–137)**.**

(a) Let f be a function defined on some interval (a, ∞). Then $\lim_{x\to\infty} f(x) = L$ means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x > N \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

(b) Let f be a function defined on some interval $(-\infty, a)$. Then $\lim_{x\to-\infty} f(x) = L$ means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x < N \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

(c) Let f be a function defined on some interval (a, ∞). Then $\lim_{x\to\infty} f(x) = \infty$ means that for every positive number M there is a corresponding positive number N such that if $x > N$, then $f(x) > M$.