## 2.5 Continuity (page 114)

**Definition 1** (continuous at a point, page 114)**.** A function $f(x)$ is *continuous at* $x = a$ (在 $x = a$ 處連續) if

$$\lim_{x \to a} f(x) = f(a).$$

We say that $f(x)$ is *discontinuous at* $x = a$ (or $f(x)$ has a *discontinuity* at $x = a$) (在 $x = a$ 處不連續) if $f(x)$ is *not* continuous at $a$.

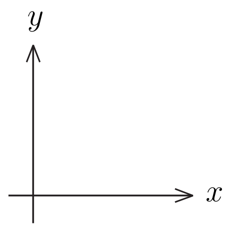


Figure 1: $f(x)$ is continuous at $x = a$.

☐ 函數 $f(x)$ 在 $x = a$ 處連續必須滿足以下三件事:

(1) $f(x)$ 在 $x = a$ 有定義, 即 $f(a)$ 存在。

(2) 極限 $\lim_{x \to a} f(x)$ 存在, 即左極限 $\lim_{x \to a^-} f(x)$ 與 $\lim_{x \to a^+} f(x)$ 存在且相等。

(3) $\lim_{x \to a} f(x) = f(a)$; 極限值等於函數值。

☐ 連續函數的另一個觀點:「極限」與「函數」可交換, 即 $\lim_{x \to a} f(x) = f(\lim_{x \to a} x) = f(a)$。
There are three types of discontinuity:

(1) *removable discontinuity* (可移不連續點): We can "redefine" the value of the function $f(x)$ at $x = a$ such that $f(x)$ is continuous at $x = a$.

(2) *infinite discontinuity* (無限不連續點).

(3) *jump discontinuity* (跳躍不連續點).

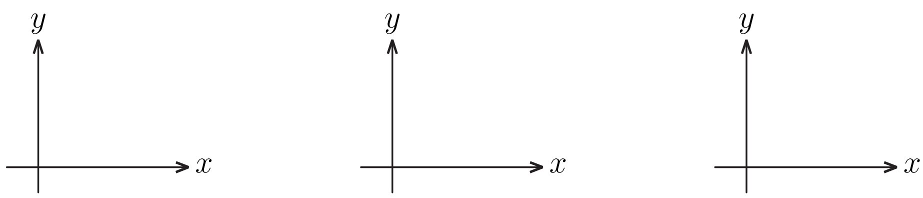


Figure 2: Three types of discontinuity.

**Definition 2** (continuous from the right (or left) (右連續與左連續), page 116)**.**

(a) A function $f(x)$ is *continuous from the right at* $x = a$ if $\lim_{x\to a^+} f(x) = f(a)$.

(b) A function $f(x)$ is *continuous from the left at* $x = a$ if $\lim_{x\to a^-} f(x) = f(a)$.

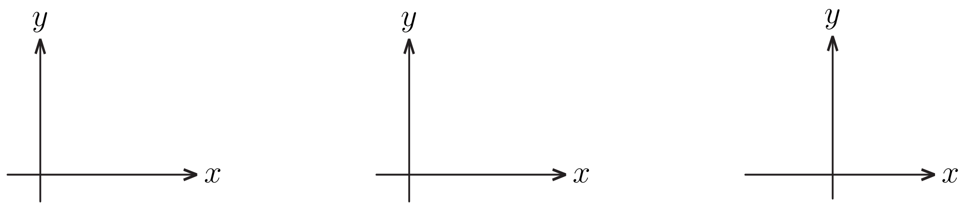


Figure 3: $f(x)$ is continuous (a) from the right; (b) from the left; (c) at endpoints.

**Example 3.** Discuss the continuity of the following functions:

$$f(x) = \frac{x^2 - x - 2}{x - 2}, \quad g(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2, \end{cases}, \quad h(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}.$$

**Solution.**

**Example 4** (同學若有興趣可想一想這個例子)**.** The *Riemann function* is defined by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q}, (p, q) = 1 \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$

Then the Riemann function is continuous at irrational numbers.

**Definition 5** (continuous on an interval, page 117)**.** A function $f(x)$ is *continuous on an interval* (在區間上連續) if it is continuous at every point in the interval. If $f(x)$ is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left.*

**Theorem 6** (properties of continuous functions, page 117)**.** *If $f(x)$ and $g(x)$ are continuous at $x = a$, and $c$ is a constant, then the following functions are also continuous $x = a$:*

(1) $(f \pm g)(x) = f(x) \pm g(x)$

(2) $cf(x)$, $cg(x)$

(3) $f(x)g(x)$

(4) $\frac{f(x)}{g(x)}$ *if* $g(a) \neq 0$.

*Proof of* (1). Since $f(x)$ and $g(x)$ are continuous at $x = a$, we have

Therefore

□

□ 連續函數經四則運算後仍爲連續函數 (除法要注意扣除分母的零點)。

**Theorem 7** (page 120)**.** *The following type of functions are continuous at every number in their domains:*

| | | |
|---|---|---|
| *polynomials* | *rational functions* | *root functions* |
| *trigonometric functions* | *inverse trigonometric functions* | *exponential functions* |
| *logarithmic functions* | | |

**Theorem 8** (page 120)**.** *If $f$ is continuous at $b$ and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$. In other words,*

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b).$$

**Example 9.** Evaluate $\lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right)$

**Solution.**

**Theorem 10** (page 121)**.** *If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then the composition function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at $a$.*

*Proof.* The function $g(x)$ is continuous at $x = a$ implies

□

☐ 連續函數的合成函數也連續。

**Theorem 11** (The Intermediate Value Theorem, 中間值定理, page 122)**.** *Suppose that $f(x)$ is continuous on the closed interval $[a, b]$ and let $N$ be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c$ in $(a, b)$ such that $f(c) = N$.*


Figure 4: The Intermediate Value Theorem.

☐ "$f(x)$ is continuous" 是必要的。

☐ "closed" interval $[a, b]$ 是必要的。

☐ 交點個數並不唯一。

## Applications of Intermediate Value Theorem

☐ 勘根定理

☐ 上山下山同時間

☐ 有限個點的平均和某點取值一樣

☐ 切蛋糕

**Example 12.** Suppose $f$ is a continuous function on $[a, b]$ and $a \leq f(x) \leq b$ for all $x \in [a, b]$. Show that there exists $c \in [a, b]$ such that $f(c) = c$.

**Solution.**