2.4 The Precise Definition of a Limit (page 104)

Definition 1 (ε-δ language, page 106)**.** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the *limit of* $f(x)$ *as* x *approaches* a *is* L, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

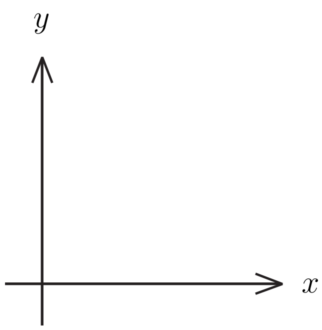

Figure 1: Limit of $f(x)$ as x approaches a is L.

- ☐ 極限的定義是“互動式”。
- ☐ δ 的找法在於“存在性”即可, 並不需要找到最佳的範圍。
- ☐ 邏輯符號: $\forall$ 代表 for all; $\exists$ 代表 exist。而 such that 數學上通常會簡記爲 s.t.。
- ☐ 搭配邏輯符號, 極限的定義可寫成: ______________________________________。

Example 2. Prove that $\lim_{x \to 1}(2x + 3) = 5$.

Solution.

- Observation: We calculate $|(2x + 3) - 5| = |2x - 2| = 2|x - 1|$. We want to find $\delta > 0$ such that

$$\text{if} \quad 0 < |x - 1| < \delta, \quad \text{then} \quad 2|x - 1| < \varepsilon.$$
$$\text{That is,} \quad \text{if} \quad 0 < |x - 1| < \delta, \quad \text{then} \quad |x - 1| < \frac{\varepsilon}{2}.$$

  This suggests that we can choose $\delta = \frac{\varepsilon}{2}$. (or smaller)

- Proof:

Example 3. Prove that $\lim_{x \to 3} x^2 = 9$.

Solution.

- Observation: We calculate $|x^2 - 9| = |x+3||x-3| < \varepsilon$. We want to find $\delta > 0$ such that

  $$\text{if} \quad 0 < |x-3| < \delta, \quad \text{then} \quad |x+3||x-3| < \varepsilon.$$

  Notice that if we can find a positive constant M such that $|x+3| < M$, then $|x+3||x-3| < M|x-3|$, and then we can make $M|x-3| < \varepsilon$ by taking $|x-3| < \frac{\varepsilon}{M} = \delta$.

  Since we are interested only in values of x that close to 3, it is reasonable to assume $|x-3| < 1$, then $|x+3| < 7$, so $M = 7$ is a choice.

- Proof:

☐ 考試作答的時候只要寫 Proof 的那段論證即可。

Example 4. Prove the Limit Sum Law: Suppose that the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.

Proof. For all $\varepsilon > 0$, since

☐

Question 5. How do we show that the limit $\lim_{x \to a} f(x)$ does not exist?

Solution. 證明極限不存在的其中一招是反證法: 假設極限存在, 記為 $\lim_{x \to a} f(x) = L$, 然後要證明「任何的」$L \in \mathbb{R}$ 都會產生矛盾。那麼怎樣才有矛盾呢? 按極限定義, 必須證明 "there exists $\varepsilon > 0$, for all $\delta > 0$, there exists $0 < |x' - a| < \delta$ s.t. $|f(x') - L| \geq \varepsilon$"。

Solution 2. 因為極限存在必唯一, 所以利用反證法: 若能設法找到「兩種數列」, 其極限值不同, 那麼就得到極限不存在。

Example 6. The *Dirichlet function* is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that $\lim_{x \to a} f(x)$ does not exist for every $a \in \mathbb{R}$.

Solution. Suppose $\lim_{x \to a} f(x) = L$. Notice that both rational numbers and irrational numbers are dense in real numbers.

<u>If $L \geq \frac{1}{2}$,</u>

<u>If $L < \frac{1}{2}$,</u>

☐ 無法正確畫出 Dirichlet 函數的圖形。

Example 7 (同學若有興趣可想一想這個例子)**.** The *Riemann function* is defined by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q}, (p, q) = 1 \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Then the limit of $f(x)$ exists as x approaches to any irrational number.

Definition 8 (Definition of left-hand limit, page 109)**.**

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a - \delta < x < a \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

Definition 9 (Definition of right-hand limit, page 109)**.**

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a < x < a + \delta \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

Definition 10 (page 112). Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every number M there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) > M.$$

☐ 無限大的意義: ______________________________

Example 11. Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Solution.

- Observation: Let M be a given positive number. We want to find a number $\delta > 0$ such that if $0 < |x| < \delta$, then $\frac{1}{x^2} > M$. Notice that

$$\frac{1}{x^2} > M \Leftrightarrow x^2 < \frac{1}{M} \Leftrightarrow |x| < \frac{1}{\sqrt{M}}.$$

  This suggests us to choose $\delta = \frac{1}{\sqrt{M}}$ (or smaller).

- Proof:

Definition 12 (page 112). Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that for every number N there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) < N.$$