2.3 Calculating Limits Using the Limit Laws, page 95

Theorem 1. *If* $\lim_{x\to a} f(x)$ *exists, then it is unique.*

Theorem 2 (Limit laws, page 95)**.** *Suppose that c is a constant and the limits* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist. Then*

(1) $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$. (Sum Law)

(2) $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$. (Difference Law)

(3) $\lim_{x\to a}(cf(x)) = c\lim_{x\to a} f(x)$. (Constant Multiple Law)

(4) $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$. (Product Law)

(5) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} g(x) \neq 0$. (Quotient Law)

The followings are some special limits:

(6) $\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$ *where* $n \in \mathbb{N}$. (Power Law)

(7) $\lim_{x\to a} c = c$.

(8) $\lim_{x\to a} x = a$.

(9) $\lim_{x\to a} x^n = a^n$ *where* $n \in \mathbb{N}$.

(10) $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ *where* $n \in \mathbb{N}$. (*If* n *is even, we assume* $a > 0$.)

(11) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ *where* $n \in \mathbb{N}$. (*If* n *is even, we assume* $\lim_{x\to a} f(x) > 0$.)

☐ 使用定理前, 必須檢查“所有條件”是否均成立。

☐ 上述定理亦適用於單側極限。

☐ 極限的四則運算可推廣至「有限個」的操作 (數學歸納法)。

Example 3. Find the limit $\lim_{x\to 1} \frac{x^2-1}{x-1}$. (比較 Section 2.1, **Example 1**.)

Solution. Let $x = 1 + \Delta x$, then $x \to 1$ is equivalent to $\Delta x \to 0$, and

Solution 2.

☐ 解法一, 利用變數變換。解法二, 要先將函數適當整理後, 再用極限定理。

Example 4. Find the limit $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$.

Solution.

☐ 遇到根號類型, 應聯想到平方差公式: $(a+b)(a-b) = a^2 - b^2$。

Example 5. Prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Solution.

☐ 若有看到絕對值, 可以試著拆絕對值後考慮單邊極限。

Theorem 6 (page 101)**.** *If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then*

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

☐ 定理的條件若改成 $f(x) < g(x)$, 取極限後仍爲「小於等於」$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$。

☐

Theorem 7 (The Squeeze Theorem (夾擠定理; 三明治定理), page 101)**.** *If*

(1) $f(x) \leq g(x) \leq h(x)$ *when x is near a (except possibly at a), and*

(2) $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$,

then $\lim_{x \to a} g(x) = L$.

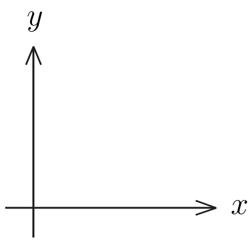

Figure 1: The Squeeze Theorem.

☐ 終極密碼。

☐ 用簡單的函數控制複雜的函數。

Example 8. Show that $\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0$.

Solution.

Example 9. Show that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$.

Solution (page 192)**.** Assume first that x lies between 0 and $\frac{\pi}{2}$. Figure 2 shows a sector of a circle with center O, central angle x, and radius 1.

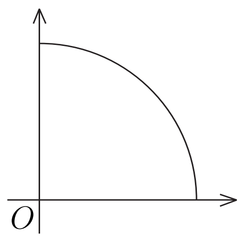

Figure 2: A sector of a circle with center O, central angle x, and radius 1.

Since "area of $\triangle OAB$" $<$ "area of sector OAB" $<$ "area of $\triangle OAC$", we have

If $-\frac{\pi}{2} < x < 0$, since $\sin x, x, \tan x$ are odd functions, we get $\tan x < x < \sin x < 0$, then

☐ 看待此極限的哲學: $\lim\limits_{\bullet\to 0} \frac{\sin\bullet}{\bullet} = 1$, 只要 $\bullet$ 放一樣的東西即可, 所以 $\lim\limits_{x\to 0} \frac{\sin 2x}{2x} = $ ___。

☐ 注意此極限是看 $\frac{\sin x}{x}$ 在「$x = 0$」附近的行爲。

Example 10. Find the limit $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2}$.

Solution. By the half-angle formula: $\sin^2\frac{x}{2} = $ __________, we get

Example 11. Find the following limits: $\lim_{x\to 0} \frac{x^3 \sin\left(\frac{1}{x}\right)}{\sin(x^2)}$.

Solution.

Solution 2.

Example 12. Is there a number a such that

$$\lim_{x\to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

Solution.

Example 13 (夾擠定理的應用)**.**

(a) Show that: If $|f(x)| \leq |g(x)|$ and $\lim_{x\to a} |g(x)| = 0$, then $\lim_{x\to a} f(x) = 0$.

(b) Show that $\lim_{x\to 0} \sin x = 0$.

(c) Show that $\lim_{x\to a} \cos x = \cos a$ and $\lim_{x\to a} \sin x = \sin a$ for $a \in \mathbb{R}$.

Solution.

Question 14. How do we show that the limit $\lim_{x\to a} f(x)$ does not exist?