2.2 The Limit of a Function, page 83

(One sided) Limit

Definition 1 (page 88)**.** We write

$$\lim_{x\to a^-} f(x) = L$$

and say the *left-hand limit of* $f(x)$ *as* x *approaches* a (or the *limit of* $f(x)$ *as* x *approaches* a *from the left*) (左極限) is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

Similarly, if we require that x be greater than a, we get "the *right-hand limit of* $f(x)$ *as* x *approaches* a (右極限) is equal to L" and we write $\lim\limits_{x\to a^+} f(x) = L$.

☐ 記號 "$x \to a^-$" 代表只考慮 $x < a$ 的部分; 而 "$x \to a^+$" 只考慮 $x > a$ 的部分。

Figure 1: Left-hand limit and right-hand limit.

Definition 2 (The limit of a function, page 83)**.** Suppose $f(x)$ is defined when x is near the number a. Then we write $\lim\limits_{x\to a} f(x) = L$ if we can make the value of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a but not equal to a.

☐ 極限 "$\lim\limits_{x\to a} f(x) = L$" 有時候會記做 "$f(x) \to L$ as $x \to a$"。

☐ 考慮極限 $\lim\limits_{x\to a} f(x)$ 時, 函數值 $f(a)$「不重要」。

☐ 極限 $\lim\limits_{x\to a} f(x)$ 是在研究 $x = a$「附近」的行爲。

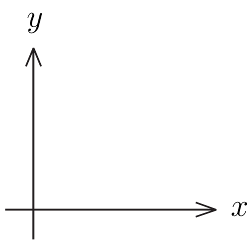

Figure 2: Limit of a function.

☐ $\lim\limits_{x\to a} f(x) = L$ 若且唯若 (if and only if) $\lim\limits_{x\to a^-} f(x) = L$ 且 $\lim\limits_{x\to a^+} f(x) = L$。

Example 3. Find the limit of the Heaviside function at $x = 0$.

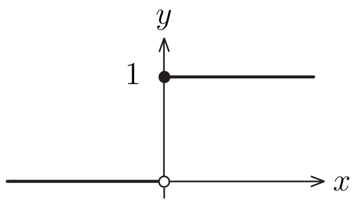

Figure 3: The Heaviside function $H(x)$.

Solution. $\lim_{x\to 0^-} H(x) =$ ____ , $\lim_{x\to 0^+} H(x) =$ ____ , $\lim_{x\to 0} H(x)$____________.

Example 4. The graph of a function $f(x)$ is shown in Figure 4. Use it to state the values (if they exist) of the following:

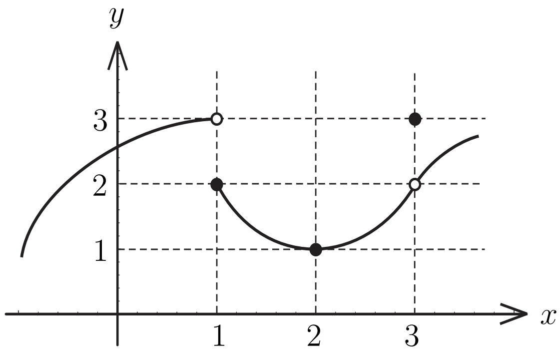

Figure 4: The graph of $f(x)$.

(a1) $\lim_{x\to 1^-} f(x)$	(b1) $\lim_{x\to 1^+} f(x)$	(c1) $\lim_{x\to 1} f(x)$	(d1) $f(1)$
(a2) $\lim_{x\to 2^-} f(x)$	(b2) $\lim_{x\to 2^+} f(x)$	(c2) $\lim_{x\to 2} f(x)$	(d2) $f(2)$
(a3) $\lim_{x\to 3^-} f(x)$	(b3) $\lim_{x\to 3^+} f(x)$	(c3) $\lim_{x\to 3} f(x)$	(d3) $f(3)$

Example 5. Observe the function $f(x) = \frac{\sin x}{x}$ and guess the value of $\lim_{x\to 0} \frac{\sin x}{x}$.

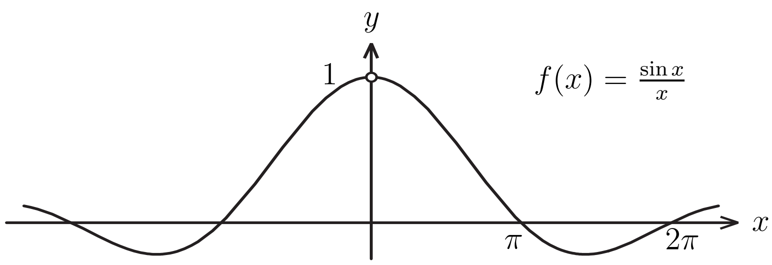

Figure 5: The graph of $f(x) = \frac{\sin x}{x}$.

Example 6. Guess the limit $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ and $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$.

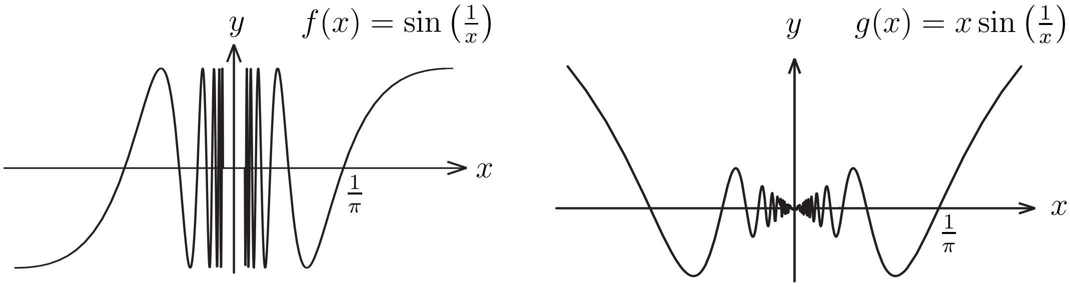

Figure 6: The graph of $f(x) = \sin\left(\frac{1}{x}\right)$ and $g(x) = x \sin\left(\frac{1}{x}\right)$.

☐ 這個例題遇到了什麼困難?

Infinite Limits

Definition 7 (page 89)**.** Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

Definition 8 (page 94)**.** Let f be a function defined on both sides of a, except possibly at a itself. Then $\lim_{x \to a} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily negative by taking x sufficiently close to a, but not equal to a.

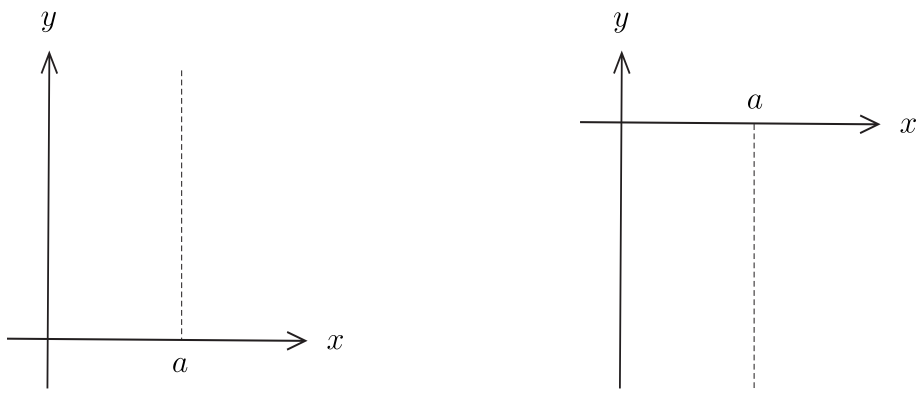

Figure 7: Infinite limit $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} f(x) = -\infty$.

☐ 極限 $\lim_{x \to a} f(x) = \infty$ 也可以寫成 "$f(x) \to \infty$ as $x \to a$."

Similar definition can be given for the one-sided infinite limits:

$$\lim_{x\to a^-} f(x) = \infty \quad \lim_{x\to a^+} f(x) = \infty \quad \lim_{x\to a^-} f(x) = -\infty \quad \lim_{x\to a^+} f(x) = -\infty.$$

Definition 9 (page 90)**.** The line $x = a$ is called a *vertical asymptote* (垂直漸近線) of the curve $y = f(x)$ if at least one of the following statement is true:

$$\begin{array}{lll} \lim_{x\to a} f(x) = \infty & \lim_{x\to a^-} f(x) = \infty & \lim_{x\to a^+} f(x) = \infty \\ \lim_{x\to a} f(x) = -\infty & \lim_{x\to a^-} f(x) = -\infty & \lim_{x\to a^+} f(x) = -\infty. \end{array}$$

Example 10.

(a) $f(x) = \tan x$ has vertical asymptotes ____________________.

(b) $f(x) = \sec x$ has vertical asymptotes ____________________.

(c) $f(x) = \frac{1}{x}$ has a vertical asymptote ______________.

(d) $f(x) = \ln x$ has a vertical asymptote ______________.