Chapter 2 Limits and Derivatives

2.1 The Tangent and Velocity Problem (page 78)

Question 1. What will we learn in the Calculus course?

The tangent problem, page 78

Example 2. Plot the parabola $f(x) = x^2$. Observe all secant lines (割線) passing through the point $P(1, f(1))$ and $Q_{\Delta x}(1+\Delta x, f(1+\Delta x))$, where $\Delta x \neq 0$ is a number close to 0.

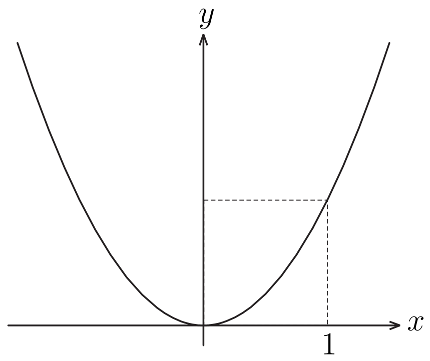

Figure 1: The parabola $f(x) = x^2$ and secant lines passing through $P(1, 1)$.

Solution. We can compute the slope of secant line $L_{PQ_{\Delta x}}$ to get

$$m_{PQ_{\Delta x}} =$$

$$=$$

So the equation of secant line $L_{PQ_{\Delta x}}$ is ________________________. When Δx is close to 0, the slope $m_{PQ_{\Delta x}}$ is close to 2. That means the family of secant lines $L_{PQ_{\Delta x}}$ is close to the line $y - 1 = 2(x - 1)$, which passes through $P(1, f(1))$ and the slope is 2.

We call $y - 1 = 2(x - 1)$ the *tangent line* (切線) of $f(x) = x^2$ at $x = 1$.

The velocity problem, page 80

☐ 汽車與機車的儀表板或自行車的碼表, 記錄里程並顯示“瞬時速度”。

☐ 棒球投手投球瞬間的速度; 網球及羽球比賽球員揮拍或殺球的球速。(大螢幕顯示)

Example 3. Suppose that a ball is dropped from the upper observation deck of Taipei 101. Find the velocity of the ball after 5 seconds.

Solution. If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation

$$s(t) = \frac{1}{2} \cdot 9.8 \cdot t^2 = 4.9t^2.$$

We can approximate the velocity at instant time $t = 5$ by computing the average velocity over the brief time interval

$$\begin{aligned}\text{average velocity} &= \frac{\text{change in position}}{\text{time elapsed}} = \frac{s(5+10^{-n}) - s(5)}{(5+10^{-n}) - 5} \\ &= \frac{4.9 \cdot ((5+10^{-n})^2 - 5^2)}{10^{-n}} = \frac{4.9 \cdot (5+10^{-n}+5)(5+10^{-n}-5)}{10^{-n}} \\ &= 4.9 \cdot (10+10^{-n}) = 49 + 4.9 \cdot 10^{-n}.\end{aligned}$$

That is,

Time interval	Average velocity (m/s)
$5 \le t \le 5.1$	49.49
$5 \le t \le 5.01$	49.049
$5 \le t \le 5.001$	49.0049
$5 \le t \le 5.0001$	49.00049
$5 \le t \le 5.00001$	49.000049

It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The *instantaneous velocity* (瞬時速度) when $t = 5$ is defined to be the limiting value of these average velocities over shorter an shorter time periods that start at $t = 5$. Thus the instantaneous velocity after 5 second is $v = 49$ m/s.

Remark 4. Time periods 10^{-n} we choose in **Example 3** are just some samples. In general, we can use Δt to represent any time interval and do the same calculation to get the average velocity form 5 to $5 + \Delta t$ is $4.9 \cdot (10 + \Delta t)$. The average velocity is becoming closer to 49 m/s as well when we shorten the time period.