1.5 Inverse Functions and Logarithms, page 55

Definition 1 (page 55)**.** A function f is called a *one-to-one function* (一對一函數) if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2.$$

☐ 一對一函數的等價敘述是: 若 $f(x_1) = f(x_2)$, 則 $x_1 = x_2$.

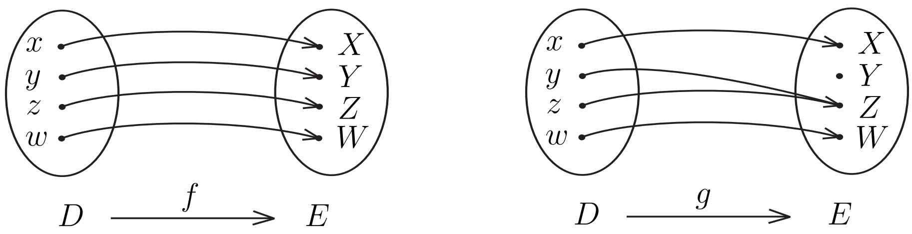

Figure 1: f is one-to-one; g is not.

Horizontal Line Test (page 56)**.** *A function is one-to-one if and only if no horizontal line intersects its graph more than once.*

☐ 判斷一個函數圖形是否爲一對一, 幾何上使用「水平線法」。

Example 2. Plot the graphs of a one-to-one and not a one-to-one function.

Figure 2: Left: a one-to-one function; Right: not a one-to-one function.

Definition 3 (page 56)**.** Let f be a one-to-one function with domain D and range E. Its *inverse function* (反函數) f^{-1} with domain E and range D is defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y \quad \text{for any } y \text{ in } E.$$

☐ $f^{-1}(x)$ 是逆運算, 不同於函數的倒數 $\frac{1}{f(x)} = (f(x))^{-1}$。

☐ 數學上常用 x 爲自變數, y 爲應變數, 所以反函數會寫成 $y = f^{-1}(x) \Leftrightarrow f(y) = x$。

☐ 函數與反函數有消去律: $f^{-1}(f(x)) = x \ \ \forall x \in D$; $f(f^{-1}(x)) = x \ \ \forall x \in E$。

Question 4. How do we find the inverse function of a one-to-one function $y = f(x)$?

(1) Solve this equation for x in terms of y (if possible).

(2) Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Example 5. Find a formula for the inverse of the function $f(x) = x^2 - x, x \geq \frac{1}{2}$.

Solution.

☐ 所有函數圖形 $y = f(x)$ 與其反函數圖形 $y = f^{-1}(x)$ 必對稱於 ______.

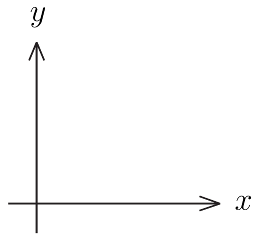

Figure 3: Symmetry of a function and its inverse function.

☐ 任何遞增 (遞減) 函數必存在反函數。(從 _____ 理解或從 __________ 看之。)

Example 6. Plot the graph $f(x) = \frac{1}{\sqrt{x}} = \frac{1}{x^{\frac{1}{2}}}$ and $g(x) = \frac{1}{x^2}, x > 0$.

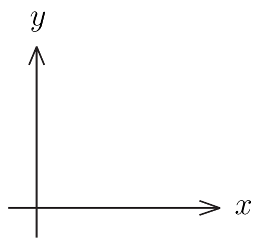

Figure 4: $f(x) = \frac{1}{\sqrt{x}}$ and $g(x) = \frac{1}{x^2}, x > 0$.

Logarithmic Functions

Definition 7 (page 59)**.** If $a > 0$ and $a \neq 1$, the exponential function $f(x) = a^x$ is either increasing or decreasing. It therefore has an inverse function $f^{-1}(x)$, which is called the *logarithmic function with base a* (以 a 爲底的對數函數) and is denoted by $\log_a x$.

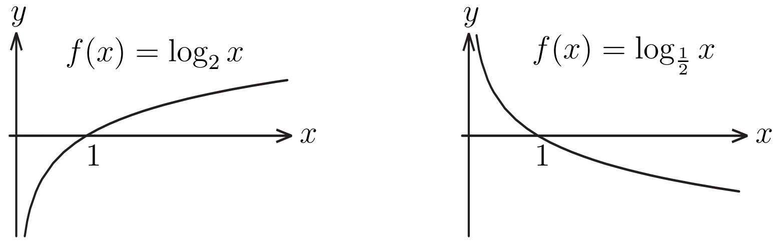

Figure 5: Logarithmic functions.

Since $y = f^{-1}(x) \Leftrightarrow f(y) = x$, we have $y = \log_a x \Leftrightarrow a^y = x$.

☐ 對所有 $x \in \mathbb{R}$, $\log_a(a^x) = x$ (由函數與反函數的消去律可看出)。

☐ 對所有 $x > 0$, $a^{\log_a x} = x$ (由函數與反函數的消去律可看出)。

Laws of Logarithms (page 59)**.** *If x and y are positive numbers, then*

(1) $\log_a(xy) = \log_a x + \log_a y$.

(2) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.

(3) $\log_a(x^r) = r\log_a x$ (*where r is any real number*).

Definition 8 (page 60)**.** The logarithm with base $\mathrm{e} \approx 2.718281828\ldots$ is called the *natural logarithm* (自然對數) and has a special notation:

$$\log_{\mathrm{e}} x = \ln x.$$

☐ $\ln x = y \Leftrightarrow \mathrm{e}^y = x$.

☐ $\ln(\mathrm{e}^x) = x$ for all $x \in \mathbb{R}$.

☐ $\mathrm{e}^{\ln x} = x$ for all $x > 0$.

☐ $\ln \mathrm{e} = 1$.

Property 9 (Change of base formula, page 62)**.** *For any positive number a ($a \neq 1$),*

$$\log_a x = \frac{\ln x}{\ln a}.$$

Inverse Trigonometric Functions

The sine function $f(x) = \sin x$ is not one-to-one, but the function $f(x) = \sin x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ is one-to-one.

Definition 10 (page 63)**.** The inverse function of this restricted sine function exists and is denoted by $\sin^{-1} x$ or arcsin x. It is called the *inverse sine function* or the *arcsine function* (反正弦函數).

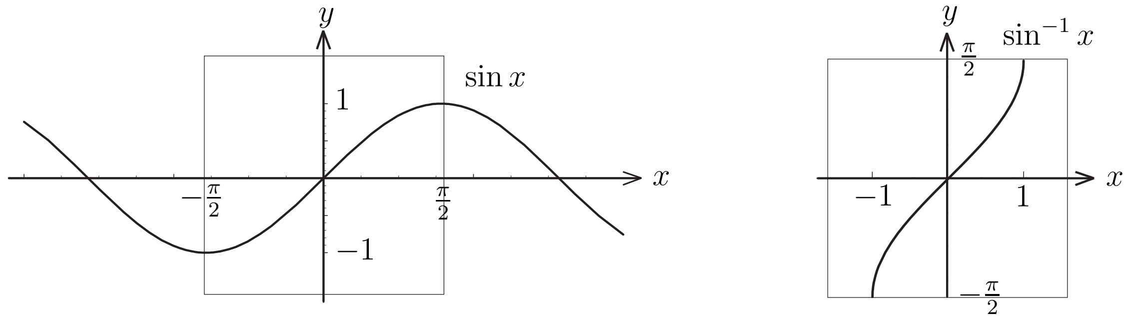

Figure 6: $\sin x$ and $\sin^{-1} x$.

☐ 特別注意 $\sin^{-1} x \neq \frac{1}{\sin x} = (\sin x)^{-1} = \csc x \neq \sin(x^{-1}) = \sin\frac{1}{x}$。

☐ 消去律得知 $\sin^{-1}(\sin x) = x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ 及 $\sin(\sin^{-1} x) = x$ for $-1 \leq x \leq 1$。

Example 11. Evaluate (a) $\sin^{-1}(\frac{\sqrt{2}}{2})$ and (b) $\tan(\arcsin\frac{2}{3})$.

Solution.

Definition 12 (反餘弦、反正切、反餘切、反正割、反餘割函數, page 66)**.**

	Restriction	Inverse function	Notation
$\cos x$	$0 \leq x \leq \pi$	*inverse cosine function*	$\cos^{-1} x$ or $\arccos x$
$\tan x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$	*inverse tangent function*	$\tan^{-1} x$ or $\arctan x$
$\cot x$	$0 < x < \pi$	*inverse cotangent function*	$\cot^{-1} x$ or $\operatorname{arccot} x$
$\sec x$	$0 \leq x < \frac{\pi}{2}$ or $\pi \leq x < \frac{3\pi}{2}$	*inverse secant function*	$\sec^{-1} x$ or $\operatorname{arcsec} x$
$\csc x$	$0 < x \leq \frac{\pi}{2}$ or $\pi < x \leq \frac{3\pi}{2}$	*inverse cosecant function*	$\csc^{-1} x$ or $\operatorname{arccsc} x$.

☐ 關於 $\operatorname{arcsec} x$ 與 $\operatorname{arccsc} x$ 的值域, 並沒有統一的限制範圍。

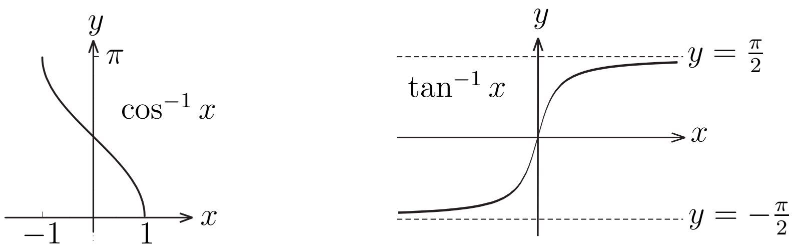

Figure 7: $\cos^{-1} x$ and $\tan^{-1} x$.

Example 13. Simplify the expression $\sin(\tan^{-1} x)$.

Solution.