## 1.2 Mathematical Models: A Catalog of Essential Functions, page 23

### Mathematical models

Why do we learn mathematics? One reason is that mathematics can help us solve problems.

$$\begin{array}{ccc} \text{Real-world problem} & \longrightarrow & \text{Mathematical model} \\ & & \downarrow \\ \text{Real-world predictions} & \longleftarrow & \text{Mathematical conclusions} \end{array}$$

### Essential functions

**Definition 1** (page 27)**.** A function $P$ is called *polynomial* (多項式) if

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0,$$

where $n$ is a nonnegative integer and the numbers $a_0, a_1, \ldots, a_n$ are constants called the *coefficients* (係數) of the polynomial. If the leading coefficient $a_n \neq 0$, then the *degree* (次數) of the polynomial is $n$.

(1) A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a *linear function*. (中學數學學會畫線性函數)

(2) A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and so it is a *quadratic function*. (中學數學學會畫二次函數)

(3) A polynomial of degree 3 is of the form $P(x) = ax^3 + bx^2 + cx + d$ and so it is a *cubic function*. (學完微積分可以知道所有的三次多項式)

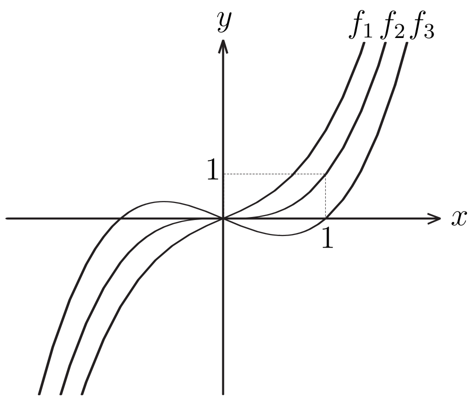


Figure 1: Some cubic functions: $f_1(x) = x^3 + x$, $f_2(x) = x^3$, $f_3(x) = x^3 - x$.

## Desmos Calculator (數學繪圖軟體)

Desmos Calculator is a free, online, mathematical software. The link is:

`https://www.desmos.com/calculator`

It is easy to use Desmos Calculator to plot the graphs of functions, equations and inequalities. More advanced features are well-designed such as polar function graphing. Users can use their Gmail account to login and save the graphs.

In calculus class, we will often use Desmos Calculator to get familiar with many concepts such as derivatives, tangent lines, polar curves, Taylor expansions, etc. Furthermore, many important graph properties such as symmetry, shift, stretching, reflection, rotation, are clearly presented by Desmos Calculator.

**Definition 2** (page 29)**.** A function of the form $f(x) = x^a$, where $a$ is a constant, is called a *power function* (冪函數).

We usually consider several cases.

(a) $a = n$, where $n$ is a positive integer *polynomial with one term* (多項式).

(b) $a = \frac{1}{n}$, where $n$ is a positive integer. *root function* (根式函數).

(c) $a = -1$. *reciprocal function* (倒數函數).

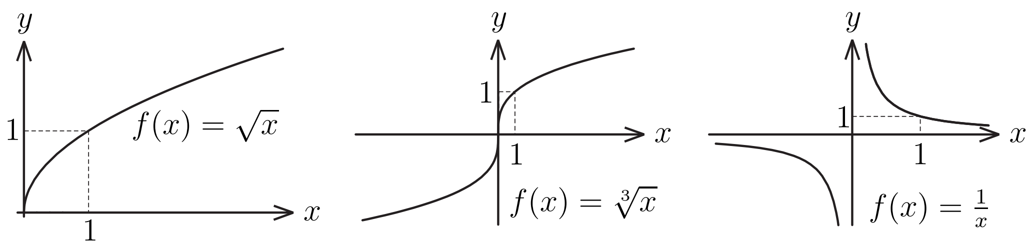


Figure 2: Graphs of root function and reciprocal function.

**Definition 3** (page 30)**.** A *rational function* (有理函數) is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)},$$

where $P$ and $Q$ are polynomials. The domain consists of all values of $x$ such that $Q(x) \neq 0$.

**Definition 4** (page 30)**.** A function $f$ is called an *algebraic function* (代數函數) if it can be construct using algebraic operations (addition, subtraction, multiplication, division, and roots) starting with polynomials.

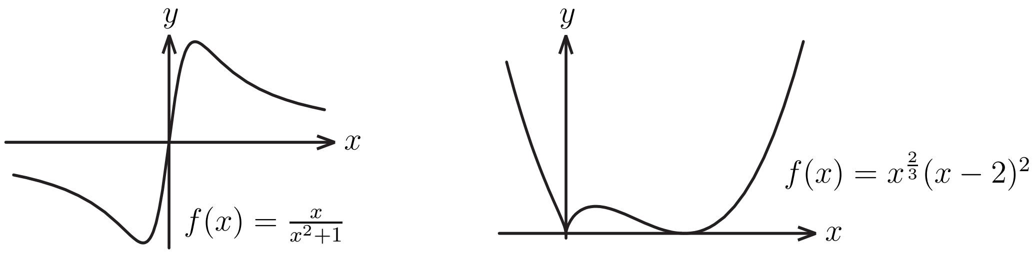


Figure 3: Graphs of rational function and algebraic function.

**Example 5.** The mass of a particle with velocity $v$ is $m = f(v) = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, where $m_0$ is the rest mass of the particle; $c = 3 \times 10^5$ km/s is the speed of light in a vacuum.

**Definition 6** (page A26)**.** Let $P(x, y)$ by any point on the terminal side of $\theta$ and let $r$ be the distance $|OP|$. Then we define *trigonometric functions* (三角函數) as:

$$\sin\theta = \frac{y}{r} \quad \cos\theta = \frac{x}{r} \quad \tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y} \quad \sec\theta = \frac{r}{x} \quad \csc\theta = \frac{r}{y}.$$

Angles can be measured in degrees or in radians (abbreviated as rad). The angle given by a complete revolution contain $360°$, which is the same as $2\pi$ rad.

There are a lot of *trigonometric identities* (三角恆等式):

☐ $\sin\theta\csc\theta = 1, \quad \cos\theta\sec\theta = 1, \quad \tan\theta\cot\theta = 1.$

☐ $\cos\theta\tan\theta = \sin\theta, \quad \sin\theta\cot\theta = \cos\theta, \quad \sin\theta\sec\theta = \tan\theta,$
$\cos\theta\csc\theta = \cot\theta, \quad \tan\theta\csc\theta = \sec\theta, \quad \cot\theta\sec\theta = \csc\theta.$

☐ $\sin^2\theta + \cos^2\theta = 1, \quad \tan^2\theta + 1 = \sec^2\theta, \quad 1 + \cot^2\theta = \csc^2\theta.$

☐ $\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta, \quad \tan(-\theta) = -\tan\theta,$
$\cot(-\theta) = -\cot\theta, \quad \sec(-\theta) = \sec\theta, \quad \csc(-\theta) = -\csc\theta.$

☐ $\sin(\theta + 2\pi) = \sin\theta, \quad \cos(\theta + 2\pi) = \cos\theta, \quad \tan(\theta + \pi) = \tan\theta,$
$\cot(\theta + \pi) = \cot\theta, \quad \sec(\theta + 2\pi) = \sec\theta, \quad \csc(\theta + 2\pi) = \csc\theta.$

☐ $\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y,$
$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y,$
$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}.$

☐ $\sin 2x = 2\sin x\cos x,$
$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x.$

☐ $\cos^2 x = \frac{1+\cos 2x}{2},$
$\sin^2 x = \frac{1-\cos 2x}{2}.$

☐ $2\sin x\cos y = \sin(x+y) + \sin(x-y)$
$2\cos x\cos y = \cos(x+y) + \cos(x-y)$
$2\sin x\sin y = -\cos(x+y) + \cos(x-y)$

**Definition 7.** A function $f(x)$ is called *bounded on an interval* $I$ (有界函數) if there exists a constant $M$ such that $|f(x)| \leq M$ for $x \in I$.

**Example 8.** $\sin x, \cos x$ are bounded functions on $\mathbb{R}$. $\sin\left(\frac{1}{x}\right)$ is bounded on $x \neq 0$. $|x|$ is not bounded function on $\mathbb{R}$.

**Definition 9.** A function $f(x)$ is called *periodic* with period $T$ (nonzero constant) (周期函數) if $f(x+T) = f(x)$ for $x$ is defined.

**Example 10.** $\sin x, \cos x, \sec x$ and $\csc x$ are periodic functions with period $2\pi$; $\tan x$ and $\cot x$ are periodic functions with period $\pi$.

**Definition 11** (page 32)**.** The *exponential function* (指數函數) are the functions of the form $f(x) = a^x$, where the basis $a$ is a positive constant.

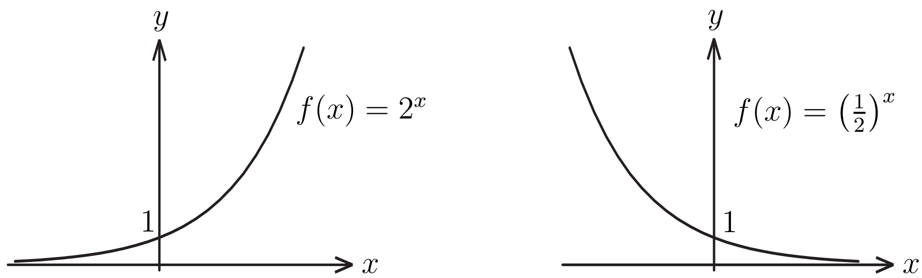


Figure 4: Exponential functions.

**Definition 12** (page 32)**.** The *logarithmic function* (對數函數) $f(x) = \log_a x$, where the base $a$ ia a positive constant, are the inverse functions of the exponential functions.

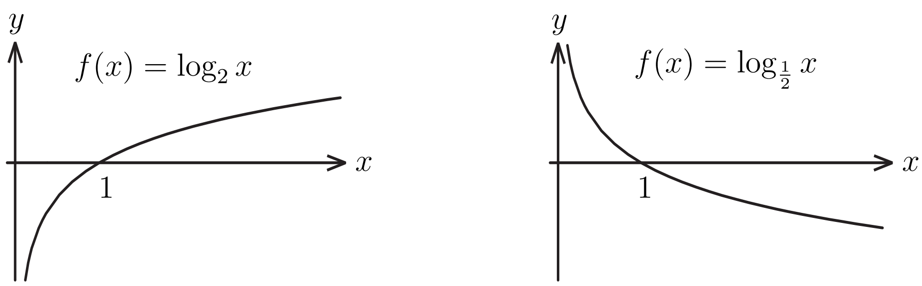


Figure 5: Logarithmic functions.