微分幾何 (二) 課程學習單　　　　　　　　　　　　　　活動 8

學號: _____　　　姓名: _____　　　你的伙伴: _____

1　單元介紹與學習目標

☐ 高斯絕妙定理與曲面論基本定理的更進一步認識。

2　高斯絕妙定理的更進一步認識

例題 1. 若 $\mathbf{x}(u,v)$ 是一個 正交參數式 (orthogonal parametrization), 也就是說 $F = 0$, 則

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)。$$

解. 因為

$$\Gamma_{11}^1 E = \tfrac{1}{2}E_u \quad \Rightarrow \Gamma_{11}^1 = \frac{E_u}{2E} \qquad \Gamma_{11}^2 G = -\tfrac{1}{2}E_v \quad \Rightarrow \Gamma_{11}^2 = -\frac{E_v}{2G}$$

$$\Gamma_{12}^1 E = \tfrac{1}{2}E_v \quad \Rightarrow \Gamma_{12}^1 = \frac{E_v}{2E} \qquad \Gamma_{12}^2 G = \tfrac{1}{2}G_u \quad \Rightarrow \Gamma_{12}^2 = \frac{G_u}{2G}$$

$$\Gamma_{22}^1 E = -\tfrac{1}{2}G_u \quad \Rightarrow \Gamma_{22}^1 = -\frac{G_u}{2E} \qquad \Gamma_{22}^2 G = \tfrac{1}{2}G_v \quad \Rightarrow \Gamma_{22}^2 = \frac{G_v}{2G}$$

所以

$$\begin{aligned}
-EK &= (\Gamma_{12}^2)_u + \Gamma_{12}^1\Gamma_{11}^2 + \Gamma_{12}^2\Gamma_{21}^2 - (\Gamma_{11}^2)_v - \Gamma_{11}^1\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2 \\
&= \left(\frac{G_u}{2G}\right)_u - \frac{E_v}{2E}\cdot\frac{E_v}{2G} + \frac{G_u}{2G}\cdot\frac{G_u}{2G} + \left(\frac{E_v}{2G}\right)_v - \frac{E_u}{2E}\cdot\frac{G_u}{2G} + \frac{E_v}{2G}\cdot\frac{G_v}{2G} \\
&= \frac{G_{uu}}{2G} - \frac{G_u^2}{2G^2} - \frac{E_v}{2E}\cdot\frac{E_v}{2G} + \frac{G_u}{2G}\cdot\frac{G_u}{2G} + \frac{E_{vv}}{2G} - \frac{E_vG_v}{2G^2} - \frac{E_u}{2E}\cdot\frac{G_u}{2G} + \frac{E_v}{2G}\cdot\frac{G_v}{2G} \\
&= \frac{G_{uu}}{2G} - \frac{E_v}{2E}\cdot\frac{E_v}{2G} - \frac{G_u}{2G}\cdot\frac{G_u}{2G} + \frac{E_{vv}}{2G} - \frac{E_u}{2E}\cdot\frac{G_u}{2G} - \frac{E_v}{2G}\cdot\frac{G_v}{2G} \\
\Rightarrow K &= -\frac{G_{uu}}{2EG} + \frac{E_v^2}{4E^2G} + \frac{G_u^2}{4EG^2} - \frac{E_{vv}}{2EG} + \frac{E_uG_u}{4E^2G} + \frac{E_vG_v}{4EG^2}。
\end{aligned}$$

另一方面,

$$-\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right) = -\frac{E_{vv}}{2EG} + \frac{E_v(E_vG + EG_v)}{4E^2G^2} - \frac{G_{uu}}{2EG} + \frac{G_u(E_uG + EG_u)}{4E^2G^2},$$

因此

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)。$$

註. 在一般參數式下高斯曲率的公式為

$$K = \frac{1}{(EG - F^2)^2}\left(\begin{vmatrix} -\frac{E_{vv}}{2} + F_{uv} - \frac{G_{uu}}{2} & \frac{E_u}{2} & F_u - \frac{E_v}{2} \\ F_v - \frac{G_u}{2} & E & F \\ \frac{G_v}{2} & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{E_v}{2} & \frac{G_u}{2} \\ \frac{E_v}{2} & E & F \\ \frac{G_u}{2} & F & G \end{vmatrix}\right)。$$

例題 **2.** 若 $\mathbf{x}(u, v)$ 是一個等溫參數式 (isothermal parametrization), 也就是說 $E = G = \lambda(u, v) > 0, F = 0$, 則

$$K = -\frac{1}{2\lambda}\Delta(\ln \lambda),$$

其中 $\Delta\varphi = \frac{\partial^2\varphi}{\partial u^2} + \frac{\partial^2\varphi}{\partial v^2}$ 是函數 φ 的 拉普拉斯 (Laplacian)。特別地, 當 $E = G = (u^2 + v^2 + c)^{-2}, F = 0$, 則 $K = 4c$。

解. 透過 例題 **1** 的公式, 得到

$$K = -\frac{1}{2\lambda}\left(\left(\frac{\lambda_v}{\lambda}\right)_v + \left(\frac{\lambda_u}{\lambda}\right)_u\right) = -\frac{1}{2\lambda}\left((\ln\lambda)_{vv} + (\ln\lambda)_{uu}\right) = -\frac{1}{2\lambda}\Delta(\ln\lambda)。$$

當 $\lambda = (u^2 + v^2 + c)^{-2}$, 則 $\ln\lambda = -2\ln(u^2 + v^2 + c)$, 並且

$$(\ln\lambda)_u = -2\cdot\frac{2u}{u^2+v^2+c} = -\frac{4u}{u^2+v^2+c} \qquad (\ln\lambda)_{uu} = -\frac{4}{u^2+v^2+c} + \frac{4u\cdot 2u}{(u^2+v^2+c)^2}$$

$$(\ln\lambda)_v = -2\cdot\frac{2v}{u^2+v^2+c} = -\frac{4v}{u^2+v^2+c} \qquad (\ln\lambda)_{vv} = -\frac{4}{u^2+v^2+c} + \frac{4v\cdot 2v}{(u^2+v^2+c)^2}$$

所以

$$K = -\frac{1}{2\lambda}\left((\ln\lambda)_{uu} + (\ln\lambda)_{vv}\right) = -\frac{1}{2\lambda}\cdot\lambda\left(-4(u^2+v^2+c) + 8u^2 - 4(u^2+v^2+c) + 8v^2\right) = 4c。$$

例題 **3.** 無法畫一張不失真的地圖。

解.

例題 **4.** 球 (sphere)、柱 (cylinder)、馬鞍 (saddle) $z = x^2 - y^2$ 彼此不可能局部保距。

解.

例題 **5.** 驗證以下兩曲面

$$\mathbf{x}(u, v) = (u \cos v, u \sin v, \ln u), u > 0$$

$$\overline{\mathbf{x}}(u, v) = (u \cos v, u \sin v, v)$$

在對應的點 $\mathbf{x}(u, v)$ 與 $\overline{\mathbf{x}}(u, v)$ 具有同樣的高斯曲率, 但是 $\overline{\mathbf{x}} \circ \mathbf{x}^{-1}$ 並非保距映射。這個例子告知: 局部保距可得高斯曲率相同, 反之不對。

解. 欲證 $\overline{\mathbf{x}} \circ \mathbf{x}^{-1}$ 並非保距映射, 需要用到以下命題 (第 231 頁, 單元 4-2 習題 2):

命題 **6.** *Let* $\varphi : S \to \bar{S}$ *be an isometry and* $\mathbf{x} : U \to S$ *a parametrization at* $p \in S$*, then* $\bar{\mathbf{x}} = \varphi \circ \mathbf{x}$ *is a parametrization at* $\varphi(p)$ *and* $E = \overline{E}, F = \overline{F}, G = \overline{G}$.

3 曲面論基本定理的更進一步認識

例題 **7.** 試證不可能存在曲面 $\mathbf{x}(u,v)$ 滿足 $E = G = 1, F = 0, e = 1, g = -1, f = 0$。

解.

例題 **8.** 是否存在曲面 $\mathbf{x}(u,v)$ 滿足 $E = 1, F = 0, G = \cos^2 u, e = \cos^2 u, f = 0, g = 1$?

解.