Chapter 16 Vector Calculus

16.1 Vector Fields, page 1068

Definition 1 (page 1069)**.**

(1) Let D be a set in $\mathbb{R}^2$. A *vector field on* $\mathbb{R}^2$ (向量場) is a map $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

(2) Let E be a subset of $\mathbb{R}^3$. A *vector field on* $\mathbb{R}^3$ is a map $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

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第十六章是向量微積分。該主題起源於物理，許多力學可以用場論描述像是電場、磁場、重力場等，這之間蘊含著某些守恆關係，利用向量微積分的語言得以詮釋。

向量場是一個映射，從一個點對應到一個向量的關係，通常會用左圖那樣的方式示意其概念。

The best way to picture a vector field is to draw the arrow representing the vector $\mathbf{F}(x, y)$ starting at the point (x, y) for a few representative points in D.

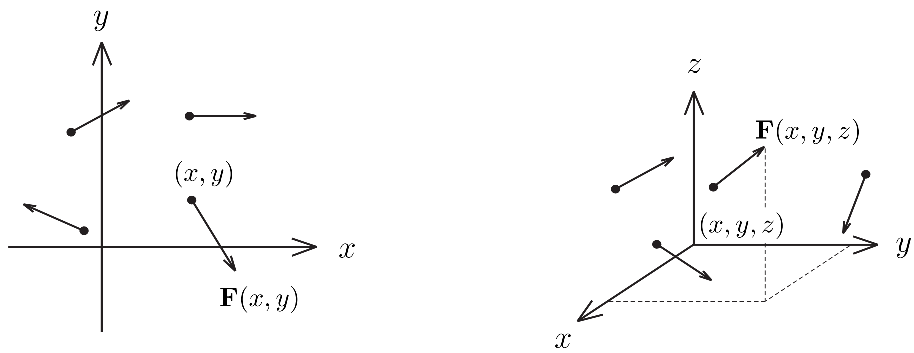

Figure 1: Vector fields on $\mathbb{R}^2$ and on $\mathbb{R}^3$.

Since $\mathbf{F}(x, y)$ is a two-dimensional vector, we can write it in terms of its *component functions* (分量函數) P and Q as follows:

這裡的學習應時時刻刻注意每個記號代表的本質是向量還是函數。

$$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} = (P(x, y), Q(x, y)).$$

Functions $P(x, y)$ and $Q(x, y)$ are called *scalar function* (純量函數) or *scalar fields*.

Example 2 (page 1070)**.** A vector field on $\mathbb{R}^2$ is defined by $\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$. Denote $\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j}$ by the position vector.

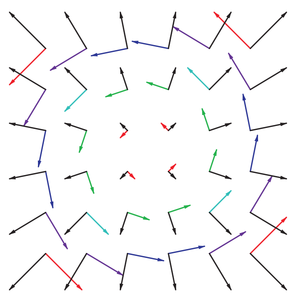

Figure 2: Vector fields $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} = (-y, x)$ and $\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j} = (x, y)$.

Remark that $\mathbf{x} \cdot \mathbf{F}(\mathbf{x}) = (x\,\mathbf{i} + y\,\mathbf{j}) \cdot (-y\,\mathbf{i} + x\,\mathbf{j}) = -xy + xy = 0$, so two vector fields are orthogonal (正交).

兩物體之間的吸引力與質量成正比，與距離平方成反比，將牛頓的重力定律用向量場的方式描述，可再將每個位置重力的方向標示出來。重力場的模型很重要，之後會繼續引申出位勢的觀念，進而證明能量守恆定律。

Example 3 (page 1071)**.** Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects with masses m and M is

$$|\mathbf{F}| = \frac{GMm}{r^2},$$

where r is the distance between the objects and G is the gravitational constant.

Let the position vector of the object with mass m be $\mathbf{x} = (x, y, z)$, then $r^2 = |\mathbf{x}|^2$. Therefore the gravitational force acting on the object at $\mathbf{x}$ is

$$\mathbf{F}(\mathbf{x}) = -\frac{GMm}{|\mathbf{x}|^2}\,\frac{\mathbf{x}}{|\mathbf{x}|} = -\frac{GMm}{|\mathbf{x}|^3}\,\mathbf{x}, \tag{1}$$

and we say the equation (1) is *gravitational field* (重力場).

電場的概念比起引力場來說，多了相斥的可能。所以描述電場時必須再研究電荷之正負號。

Example 4 (page 1072)**.** Suppose an electric charge Q is located at the origin. According to Coulomb's Law, the electric force $\mathbf{F}(\mathbf{x})$ (or electric field 電場) exerted by this charge on a charge q located at a point (x, y, z) with position vector $\mathbf{x} = (x, y, z)$ is

$$\mathbf{F}(\mathbf{x}) = \frac{\varepsilon Qq}{|\mathbf{x}|^2}\,\frac{\mathbf{x}}{|\mathbf{x}|} = \frac{\varepsilon Qq}{|\mathbf{x}|^3}\,\mathbf{x},$$

where ε is a constant. For like charges, we have $Qq > 0$ and the force is repulsive; for unlike charges, we have $Qq < 0$ and the force is attractive.

Instead of considering the electric force $\mathbf{F}$, physicists often consider the force per unit charge (電場強度):

$$\mathbf{E}(\mathbf{x}) = \frac{1}{q}\,\mathbf{F}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\,\mathbf{x}.$$

Gradient Fields (梯度場), page 1072

給定多變數函數，每一個點都可以對應到函數在該點的梯度進而造出梯度向量場，由此延伸出保守向量場的定義：若一個向量場正好與某個函數的梯度一致。而相應的函數稱爲位勢函數。

Recall that for a smooth function $f(x, y)$, the gradient ∇f, or grad f, is defined by

$$\nabla f(x, y) = \operatorname{grad} f = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}.$$

Likewise, if $f(x, y, z)$ is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by $\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}$.

Definition 5 (page 1072)**.**

(a) For a scalar function f, we say ∇f is a *gradient vector field* (梯度向量場).

(b) A vector field $\mathbf{F}$ is called a *conservative vector field* (保守向量場) if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a *potential function* (位勢函數) for $\mathbf{F}$.

Not all vector fields are conservative, but such fields do arise frequently in physics.

Example 6 (page 1073)**.** The gravitational field **F** is conservative because if we define a function

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}},$$

then

qLgmuKvQmRO

重力場是一個保守向量場，這個例子將確實找到位勢函數並驗證它的梯度與重力場一致。

Example 7. Let $f(x, y)$ be a smooth function, then the gradient vector field $\nabla f(x, y)$ is perpendicular to the level curves $f(x, y) = k$.

這裡再次複習「函數的梯度向量場與函數的等高線互相垂直」之概念。

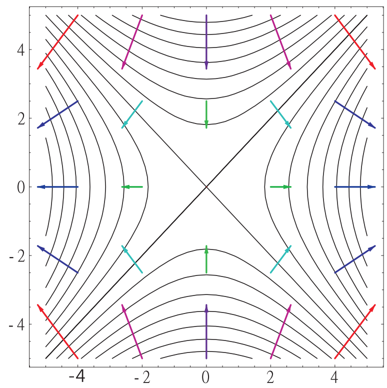

Figure 3: Level sets of $f(x, y) = x^2 - y^2$ and the gradient field $\nabla f = 2x\,\mathbf{i} - 2y\,\mathbf{j}$.

In general, all conservative vector vector field **F** is perpendicular to the level sets of its potential function f.

16.2 Line Integrals, page 1075

Line Integrals of Scalar Functions (第一類曲線積分), page 1075

HvZ-UovGbN8

Suppose that a smooth plane curve C is given by the parametric equations $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}, a \le t \le b$. Recall that a curve is smooth means that $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$. We will define an integral over a curve C.

線積分依照屬性區分爲兩類。第一類線積分的意思是: 給定一條曲線，以及定義在曲線上的函數，該如何計算函數沿著曲線的積分。第一類線積分的建構方式主要是從曲線弧長的建構當中，選定樣本點之後，順勢將這個樣本點代入函數值，最後再取和求極限而得。

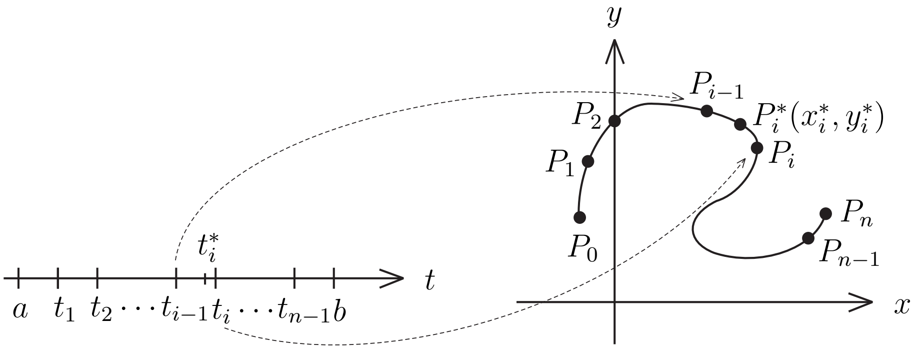

Figure 1: Line integrals of scalar functions.

(a) We divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width. We let $x_i = x(t_i)$ and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$.

(b) Choose any point $P_i(x_i^*, y_i^*)$ (corresponding to $t_i^* \in [t_{i-1}, t_i]$) in the i-th subarc.

(c) If $f(x, y)$ is any function of two variables whose domain includes the curve C, we form the sum (similar to a Riemann sum) $\sum\limits_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$.

(d) Define the *line integral of* $f(x, y)$ *along* C (函數 $f(x, y)$ 沿曲線 C 的線積分) is

$$\int_C f(x, y)\,\mathrm{d}s = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

if this limit exists.

Recall that the arc length function of C is $s(t) = \int_a^t \sqrt{(x'(u))^2 + (y'(u))^2}\,\mathrm{d}u$, and it implies $\mathrm{d}s = \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t$, so the line integral has the following expression:

$$\int_C f(x, y)\,\mathrm{d}s = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t.$$

在將第一類線積分轉變成可以計算的積分公式時，牽涉到曲線的表達。(接下頁)

Well-defined problem: (copy from Wikipedia) In mathematics, an expression is called *well-defined* (良好定義的) if its definition assigns it a unique interpretation or value. For example, *a function is well-defined* if it gives the same result when the representation of the input is changed without changing the value of the input. That is, if $f(x)$ is a well-defined function defined on $\mathbb{R}$, then $f(0.5)$ must be equal to $f(\frac{1}{2})$, and $f(1)$ must be equal to $f(0.\bar{9})$.

Suppose that $\mathbf{r}(v) = x(v)\,\mathbf{i} + y(v)\,\mathbf{j}, c \le v \le d$ is another parametrization of the plane curve C. To show the line integral is well-defined, we have to check that

因爲曲線有各種可能的表達式，在數學上還必須驗證的是： 這樣的公式與參數式的選取無關。

$$\int_C f(x,y)\,\mathrm{d}s = \int_c^d f(x(v), y(v))\sqrt{(x'(v))^2 + (y'(v))^2}\,\mathrm{d}v.$$

Check: This is because the arc length is $s(v) = \int_a^v \sqrt{(x'(u))^2 + (y'(u))^2}\,\mathrm{d}u$, and it implies $\mathrm{d}s = \sqrt{(x'(v))^2 + (y'(v))^2}\,\mathrm{d}v$.

The geometric meaning of the line integrals is to compute the area of one side of the "fence" or "curtain," whose base is C and whose "height" at (x, y) is $f(x, y)$.

qSVVaISYces

☐ $\mathrm{d}s$ 是弧長參數, 爲正量, 給定曲線參數式後, 上、下限代入終點與起點。

☐ 第一類曲線積分與「路徑的方向無關」, 所以化爲定積分時, 下限總是小於上限。

第一類曲線積分的應用有很多，比方說考慮密度不均的線圈， 將密度函數線積分而得線圈之質量。 若是平面曲線， 將函數大小呈現在 z 分量下得到函數的圖形，其形狀像是一個屏風，則將函數對於曲線進行第一類線積分將得到屏風的表面積；跑步的時候，在不同階段下熱量的消耗也不同，將熱量函數對於路徑積分而得消耗的總熱量。

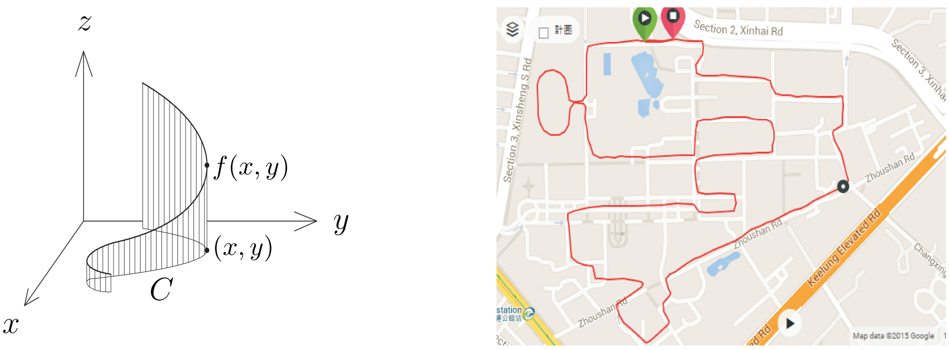

Figure 2: Line integrals: The area of the fence. The total calories of running.

Example 1 (page 1076). Evaluate $\int_C (2 + x^2 y)\,\mathrm{d}s$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

例題示範第一類線積分，先將曲線參數式，把函數替換成曲線參數式的符號，配上線元 $\mathrm{d}s$，積分時下限代入小的數，上限代入大的數。

Solution.

We can define the line integrals on *piecewise smooth curve* (分段光滑曲線) C, which is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, and the initial point of C_{i+1} is the terminal point of C_i. The line integral of $f(x, y)$ along C as the sum of the integrals of f along each of the smooth pieces of C:

f2Dwlq0Vd0M

第一類曲線積分可用在分段光滑曲線的情況，若曲線只有有限個點無法定義切向量，這件事並不影響積分值。

$$\int_C f(x,y)\,\mathrm{d}s = \int_{C_1} f(x,y)\,\mathrm{d}s + \int_{C_2} f(x,y)\,\mathrm{d}s + \cdots + \int_{C_n} f(x,y)\,\mathrm{d}s.$$

例題示範如何計算對於分段光滑曲線計算第一類線積分，其實就是分段處理再相加即可。

Example 2 (page 1077)**.** Evaluate $\int_C 2x\,\mathrm{d}s$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.

Solution.

B74qAVlm0js

將密度函數對於曲線積分得到曲線之質量，若將密度函數再乘以 x 或 y 積分之後除掉質量則爲質心的概念。

Any physical interpretation of $\int_C f(x,y)\,\mathrm{d}s$ depends on the physical interpretation of the function f. For example, we define the *mass* (質量) m of the wire:

$$m = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*, y_i^*)\Delta s_i = \int_C \rho(x,y)\,\mathrm{d}s.$$

The *center of mass* (質心) of the wire with density function $\rho(x,y)$ is

$$(\bar{x}, \bar{y}) = \left(\frac{1}{m}\int_C x\rho(x,y)\,\mathrm{d}s, \frac{1}{m}\int_C y\rho(x,y)\,\mathrm{d}s\right).$$

例題示範密度不均之線圈的質心。

Example 3 (page 1077)**.** A wire takes the shape of the semicircle $x^2 + y^2 = 1, y \geq 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

Solution.

Definition 4 (page 1066)**.**

(a) Define *line integrals of* f *along* C *with respect to* x (f 沿 C 對於 x 的線積分):

$$\int_C f(x,y)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i = \int_a^b f(x(t),y(t))x'(t)\,\mathrm{d}t.$$

(b) Define *line integrals of* f *along* C *with respect to* y (f 沿 C 對於 y 的線積分):

$$\int_C f(x,y)\,\mathrm{d}y = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta y_i = \int_a^b f(x(t),y(t))y'(t)\,\mathrm{d}t.$$

(c) The *line integral of* f *along* C *with respect to arc length* is:

$$\int_C f(x,y)\,\mathrm{d}s = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i = \int_a^b f(x(t),y(t))\sqrt{(x'(t))^2+(y'(t))^2}\,\mathrm{d}t.$$

wI_pijMjFKQ

函數沿著曲線對於 x 或 y 的線積分的定義是爲了要引進第二類曲線積分而準備。特別注意 $\mathrm{d}x$ 或 $\mathrm{d}y$ 是有方向性的，和坐標軸同向的時候爲正， 反向的時候爲負，所以會有正負抵消的效果。而 $\mathrm{d}s$ 是線元，所以不論曲線所在位置爲何、如何行走，線元都是正的。

□ 函數 f 沿 C 對於 x 的線積分, 因爲 $\mathrm{d}x$ 可能正可能負, 因此積分與路徑的方向有關。

It frequently happens that line integrals with respect to x and y occur together. When this happens, it's customary to abbreviate by writing

$$\int_C P(x,y)\,\mathrm{d}x + \int_C Q(x,y)\,\mathrm{d}y = \int_C P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y.$$

對於同一條曲線，有可能要同時計算函數沿曲線對於 x 或 y 的積分，這時積分符號可以只留最前面的記號，並不會引起混淆。

Example 5. Evaluate $\int_C y^2\,\mathrm{d}x + x\,\mathrm{d}y$, where

(a) $C = C_1$ is the line segment from $(-3,-2)$ to $(0,1)$.

(b) $C = C_2$ is the arc of the parabola $x = 1 - y^2$ from $(-3,-2)$ to $(0,1)$.

例題示範如何計算函數沿著曲線對於 x 或 y 的積分。特別注意積分的下限與下限，它是遵照起點與終點而設定。

Solution.

Line Integrals in Space, page 1080

-kIx2BM33Uo

上述所有概念都可以推廣至三度空間中的曲線，基本上就是增加 z 分量的相關資訊於所有討論中。

Suppose that C is a smooth space curve given by the parametric equation $x = x(t), y = y(t), z = z(t), a \le t \le b$ or vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$. If $f(x, y, z)$ is a smooth function that is continuous on some region containing C, then we define the *line integral of f along C* as

$$\int_C f(x, y, z)\,\mathrm{d}s = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i$$
$$= \int_a^b f(x(t), y(t), z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))}\,\mathrm{d}t = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,\mathrm{d}t.$$

Line integrals along C with respect to x, y, and z can also be defined. For example,

$$\int_C f(x, y, z)\,\mathrm{d}z = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta z_i = \int_a^b f(x(t), y(t), z(t))z'(t)\,\mathrm{d}t.$$

Therefore, we evaluate integrals of the form

$$\int_C P(x, y, z)\,\mathrm{d}x + Q(x, y, z)\,\mathrm{d}y + R(x, y, z)\,\mathrm{d}z$$

by expressing everything $(x, y, z, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ in terms of the parameter t.

Line Integrals of Vector Fields (第二類曲線積分), page 1082

現在要正式介紹第二類曲線積分，它是要研究分布在空間中的向量場如何作用在質點上，造成運行軌跡時所作的功。 同樣利用分割樣本點取和求極限的過程設法把積分式表達出來。特別注意在加總的階段， 根據作功的概念， 是在計算力對於質點運行的方向之有效力乘上運行的距離，所以在分割後的每一小段， 利用曲線在樣本點上的力與單位切向量兩者內積， 再乘上小段弧長，加總取極限而得第二類曲線積分。

Suppose that $\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$ is a continuous force field on $\mathbb{R}^3$ such as gravitational field, electric force field, etc. We want to *compute the work* (功) *done by this force* in moving a particle along a smooth curve C.

(a) We divide C into subarcs $P_{i-1}P_i$ with lengths Δs_i by dividing the parameter interval $[a, b]$ into subintervals of equal width.

(b) Choose $P_i^*(x_i^*, y_i^*, z_i^*)$ on the i-th subarc corresponding to t_i^*.

(c) If Δs_i is small, then as the particle moves from P_{i-1} to P_i along the curve, it proceeds approximately in the direction $\mathbf{T}(t_i^*)$, the *unit* tangent vector at P_i^*. Thus the total work done by the force $\mathbf{F}$ in moving the particle along C is approximately $\sum_{i=1}^{n} \mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(x_i^*, y_i^*, z_i^*)\Delta s_i$.

(d) When n tends to infinity, we define the *work* (功) W done by the force field $\mathbf{F}$:

$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z)\,\mathrm{d}s = \int_C \mathbf{F} \cdot \mathbf{T}\,\mathrm{d}s. \qquad (2)$$

Hence, *work is the line integral with respect to arc length of the tangential component of the force.*

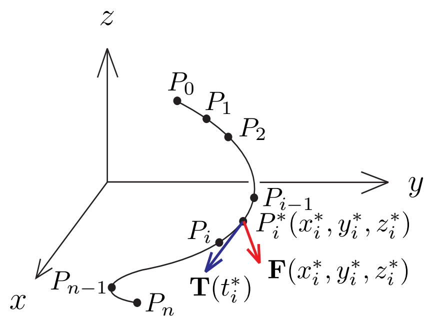

Figure 3: Line integrals of vector fields.

If the curve C is given by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, then we get the unit tangent vector is $\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, so we can rewrite (2) in the form

這裡推得第二類曲線積分的另一個表達式, 由於單位切向量必須除掉切向量的長度, 另一方面, 弧長轉換式中又帶有切向量的長度, 所以兩者相消後公式變得簡單。

$$\begin{aligned} W &= \int_C \left(\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right) |\mathbf{r}'(t)|\, \mathrm{d}t = \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, \mathrm{d}t \\ &= \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F}(x(t), y(t), z(t)) \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}. \end{aligned}$$

Definition 6 (page 1082)**.** Let $\mathbf{F}$ be a continuous vector field on a smooth curve C given by a vector function $\mathbf{r}(t), a \le t \le b$. Then the *line integral of* $\mathbf{F}$ *along* C is

DhMFx40jtRk

$$\int_C \mathbf{F} \cdot \mathbf{T}\, \mathrm{d}s = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, \mathrm{d}t.$$

這裡給予第二類曲線積分的正式定義。注意到第二類曲線積分與積分路徑的方向有關係。

- ☐ 第二類曲線積分與積分路徑的方向有關 (方向相反時, 積分值變號)。
- ☐ 積分路徑的方向會用 C 與 $-C$ 表示, 而 $\int_{-C} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$。

Connection between line integrals of vector fields and line integral of scalar functions, page 1084

Suppose that the vector field $\mathbf{F}$ on $\mathbb{R}^3$ is given in component form by the equation $\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$. Then

第二類曲線積分若改用分量的方式表達, 其結果會是函數沿著曲線對於 x, y 或 z 的線積分之加總。

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, \mathrm{d}t = \int_a^b (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot (x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k})\, \mathrm{d}t \\ &= \int_a^b (P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t))\, \mathrm{d}t \\ &= \int_C P(x, y, z)\, \mathrm{d}x + Q(x, y, z)\, \mathrm{d}y + R(x, y, z)\, \mathrm{d}z. \end{aligned}$$

例題示範第二類曲線積分的計算。第二類曲線積分具有相當豐富的結構與內容，接下來的兩單元要繼續針對第二類曲線積分的性質延伸。

Example 7. Find the work done by the force field $\mathbf{F}(x, y) = -y^2\,\mathbf{i} + x^2\,\mathbf{j}$ in moving a particle along arc of the circle $x^2 + y^2 = 2$ traversed counterclockwise from $(\sqrt{2}, 0)$ to $(-\sqrt{2}, 0)$.

Solution.

16.3 The Fundamental Theorem for Line Integrals, page 1087

Recall that Part 2 of the Fundamental Theorem of Calculus is

$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a), \tag{3}$$

where $F'(x)$ is continuous on $[a, b]$. We also called equation (3) the Net Change Theorem: The integral of a rate of change is the net change.

Here we will introduce the Fundamental Theorem for line integrals, where we think of the gradient vector ∇f of a function f as a sort of derivative of f.

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微積分基本定理是告知單變數函數的微分與積分是互逆的運算。 而線積分基本定理是在觀察多變數函數的微分與積分之間的關係。多變數函數牽涉到的微分變成梯度向量，而積分的範圍變成曲線，所以線積分基本定理的結果是：可微分函數的梯度向量場沿著曲線的第二類線積分的結果，會是終點函數值與起點函數值相減。

從證明的過程中也可以很清楚地看到線積分基本定理的本質是微積分基本定理，也因此，相關的正負號效應也都遵照當初微積分基本定理的結果。

Theorem 1 (page 1087)**.** *Let C be a smooth curve given by the vector function $\mathbf{r}(t), a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then*

$$\int_C \nabla f \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

☐ 若向量場來自於函數的梯度 (保守向量場)，則線積分的值爲兩端點函數值的差。

☐ 第二類曲線積分的路徑有方向性，即 $\int_C \nabla f \cdot \mathrm{d}\mathbf{r} = -\int_{-C} \nabla f \cdot \mathrm{d}\mathbf{r}$。

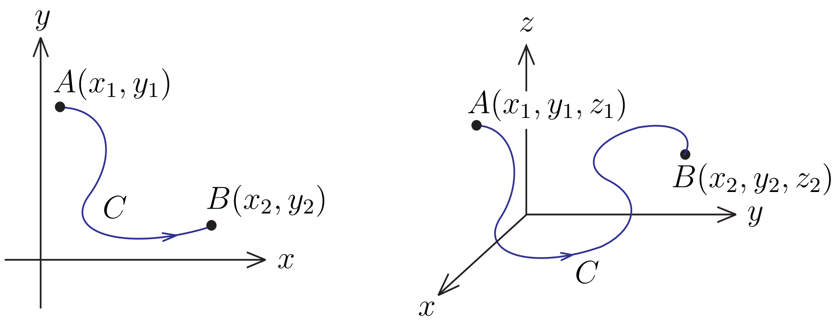

Figure 1: The fundamental theorem for line integrals.

Proof. By the Chain Rule and the Fundamental Theorem of Calculus, we have

$$\begin{aligned}\int_C \nabla f \cdot \mathrm{d}\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t = \int_a^b \left(\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}\right)\mathrm{d}t \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{r}(t))\,\mathrm{d}t = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).\end{aligned}$$

☐

Theorem 1 is also true for piecewise smooth curves. This can be seen by subdividing C into a finite number of smooth curves and adding the resulting integrals.

這裡以重力場爲例，因爲重力場是某個函數的梯度向量場，於是線積分基本定理告知，第二類曲線積分的結果會是兩端函數值相減。這個例題中並沒有明確告知曲線的走法，換言之，這個結果是與曲線的路徑無關，這件事與物理上所述位能差只與起點與終點的位置有關。

Example 2 (page 1088)**.** Find the work done by the gravitational field $\mathbf{F}(\mathbf{x}) = -\frac{GMm}{|\mathbf{x}|^3}\mathbf{x}$ in moving a particle with mass m from the point $(3, 4, 12)$ to the point $(2, 2, 0)$ along a piecewise smooth curve C.

Solution.

Independence of Path, page 1088

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Suppose that C_1 and C_2 are two piecewise smooth curves, which are also called *paths* (路徑) with have the same initial point A and terminal point B.

由重力場的例子知道有些場的第二類曲線積分與路徑選取無關，只與起點終點有關，所以特別把這個性質獨立寫成一個定義，之後要研究的是：除了保守向量場外，到底有哪些向量場也滿足積分與路徑選取無關。

Definition 3 (page 1088)**.** Suppose that $\mathbf{F}$ is a continuous vector field with domain D. We say the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is *independent of path* (積分和路徑選取無關) if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

for *any* two path C_1 and C_2 in D with the same initial and terminal points.

- In general vector field $\mathbf{F}$, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. (See 16.2, **Example 5**).
- For *conservative vector field* $\mathbf{F} = \nabla f$, the Fundamental Theorem for line integrals tells us $\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$.
- The following discussion will say that the *only* vector fields that are independent of path are conservative vector fields.

☐ 保守向量場 ⇒ 第二類曲線積分與路徑選取無關。

因爲曲線參數式是將區間 $t \in [a, b]$ 對應到平面或空間上的點 $\mathbf{r}(t)$，若要呈現封閉曲線，必須要求區間上的兩端點 a 與 b 對到平面或空間中同樣的位置。

Definition 4 (page 1089)**.** A curve is called *closed* (封閉曲線) if its terminal point coincides with its initial point, that is, $\mathbf{r}(b) = \mathbf{r}(a)$.

Figure 2: Closed curve (left) and non-closed curve (right).

Theorem 5 (page 1089)**.** *The line integral* $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ *is independent of path in* D *if and only if* $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ *for every closed path* C *in* D.

該定理表明第二類線積分若與路徑選取無關，則它等價於任何封閉曲線上的線積分值爲零。

Proof. ($\Rightarrow$) We choose any two points A and B on C and regard C as being composed of the path C_1 from A to B followed by the path C_2 from B to A. Then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1 \cup C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0.$$

($\Leftarrow$) For any paths C_1 and C_2 from A to B in D, we define C to be the curve consisting of C_1 followed by $-C_2$. Then we get

$$0 = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

and hence $\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$. ☐

☐ 第二類曲線積分與路徑無關 $\Leftrightarrow$ 封閉曲線上的第二類曲線積分爲零。

☐ 在保守向量場 (例如重力場) 將一物沿封閉曲線做功爲零。

Definition 6 (page 1089)**.**

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(a) A domain D is *open* (開集合) if for every point P in D, there is a disk with center P that lies entirely in D. (D doesn't contain any of its boundary points.)

(b) A domain D is *path connected* (路徑連通) if any two points in D can be joined by a path that lies in D.

除了上述的等價敘述外，接下來要探討第二類線積分與路徑選取無關與位勢函數的關係。以下的結果將牽涉到區域的一些特性，所以必須講清楚區域的屬性。初學階段可以先用幾個示意圖去感受每個定義最大的差別。

Figure 3: Open set; non-open set; path connected region; non-path connected region.

該定理必須注意一些關鍵字，向量場的連續性是爲了第二類線積分有意義；設定區域是開集合的用意是定理證明的過程中要用到的手法：在終點附近可以上下左右推出一個小範圍以直線相連；區域是路徑連通是要確定討論的兩點能夠用一條曲線在範圍內相連。位勢函數的找法就是先隨便取一條路徑積分而得，因爲積分與路徑選取無關，所以函數值可以確定，根據這個函數，確實驗證其梯度向量與原向量場一致。

這裡必須強調的是，定理的敘述從路徑選取無關推出向量場是保守向量場是單向的敘述，而保守向量場是否積分與路徑選取無關這個反向的敘述是否正確，是後續要討論的問題。

Theorem 7 (page 1089)**.** *Suppose* $\mathbf{F}$ *is a vector field that is continuous on an open, path connected region* D*. If* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *is independent of path in* D*, then* $\mathbf{F}$ *is a conservative vector field on* D*; that is, there exists a function* f *such that* $\nabla f = \mathbf{F}$*.*

Proof. Let $A(a,b)$ be a fixed point in D. We construct the potential function f by

$$f(x,y) = \int_{(a,b)}^{(x,y)} \mathbf{F}\cdot d\mathbf{r}$$

for $(x,y)\in D$. Since $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path, the function is well-defined. Now we will show that $\nabla f = \mathbf{F}$:

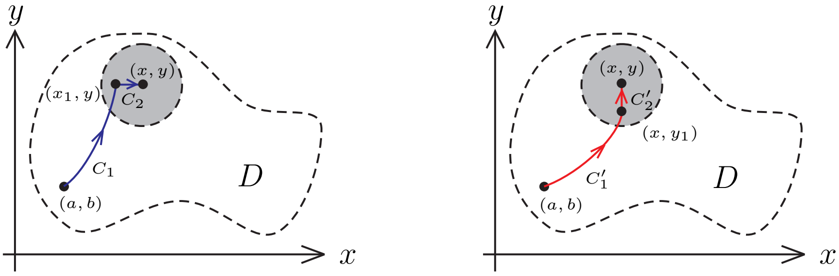

Figure 4: Choose suitable paths to prove $\nabla f = \mathbf{F}$.

Since D is open, there exists a disk contained in D with center (x,y). Choose any point (x_1,y) in the disk with $x_1 < x$ and let C consists of any path C_1 from (a,b) to (x_1,y) followed by the horizontal line segment C_2 from (x_1,y) to (x,y). Then

$$f(x,y) = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{(a,b)}^{(x_1,y)} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}.$$

Notice that the first of these integrals does *not* depend on x, so

$$\frac{\partial}{\partial x} f(x,y) = 0 + \frac{\partial}{\partial x}\int_{C_2} \mathbf{F}\cdot d\mathbf{r}. \tag{4}$$

Consider $C_2 : \mathbf{r}(t) = t\,\mathbf{i} + y\,\mathbf{j}$, where t from x_1 to x, then $\mathbf{r}'(t) = 1\,\mathbf{i} + 0\,\mathbf{j}$ and $\mathbf{F}(t) = P(t,y)\,\mathbf{i} + Q(t,y)\,\mathbf{j}$. Thus (4) gives

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \frac{\partial}{\partial x}\int_{x_1}^{x} P(t,y)\,dt = P(x,y).$$

Similarly, using a vertical line segment, consider $C = C_1' \cup C_2'$, $C_2' : \mathbf{r}(t) = x\,\mathbf{i} + t\,\mathbf{j}$, where t from y_1 to y, then $\mathbf{r}'(t) = 0\,\mathbf{i} + 1\,\mathbf{j}$ and $\mathbf{F}(t) = P(x,t)\,\mathbf{i} + Q(x,t)\,\mathbf{j}$. We have

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}\int_{C_2'} \mathbf{F}\cdot d\mathbf{r} = \frac{\partial}{\partial y}\int_{y_1}^{y} Q(x,t)\,dt = Q(x,y).$$

Therefore, we know that

$$\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j} = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} = \nabla f.$$

□

□ 由 **Theorem 7** 知：第二類曲線積分與路徑選取無關 ⇒ 保守向量場。

Next, we will determine whether or not a vector field $\mathbf{F}$ is conservative. Suppose that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is conservative, where P and Q have continuous first order partial derivatives. Then there is a function f such that $\mathbf{F} = \nabla f$, that is, $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$. By Clairaut's Theorem, we know

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若將向量場的分量寫出， 而且分量函數是一次微分連續時， 由函數二次微分可交換這件事，可以得到向量場是保守向量場的必要條件。

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}.$$

Hence $Q_x = P_y$ is a necessary condition (必要條件) that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is conservative.

Theorem 8 (page 1090)**.** *If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where P and Q have continuous first order partial derivatives on a domain D, then throughout D we have*

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

The condition $Q_x = P_y$ is a sufficient condition *for a simply connected region.*

$Q_x = P_y$ 這個條件，必須加上區域是「單連通」的時候，逆敘述才會成立。爲此，必須補充說明單連通的意義。

Definition 9 (page 1090)**.**

(a) We say C is a *simple curve* (簡單曲線) if it doesn't intersect itself anywhere between its endpoints. ($\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ when $a < t_1 < t_2 < b$).

簡單曲線要描述的是曲線不自交。而單連通的概念除了要求是路徑連通之外，還要強調這個區域的內部是沒有「洞」。

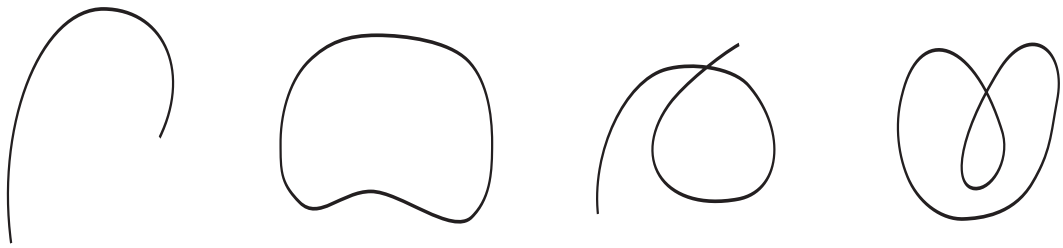

Figure 5: (Left to right) Simple, not closed; simple closed; not simple, not closed; not simple, closed.

(b) D is a *simply connected region* (單連通區域) in a plane if it is path connected and every simple closed curve in D enclosed only points that are in D.

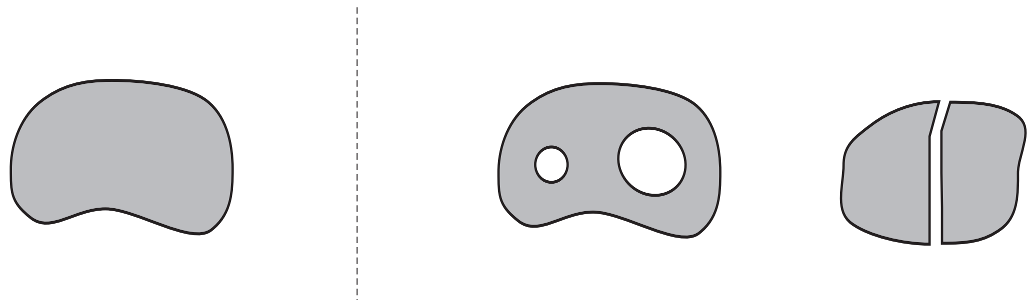

Figure 6: Simply connected region; non simply connected region.

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這個定理的重點是若討論的區域是單連通，而且定義在區域上的向量場一次微分且連續的時候，$Q_x = P_y$ 的條件才可以反推而得向量場是保守向量場。

Theorem 10 (page 1091)**.** *Let $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and*

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \quad \textit{throughout} \quad D,$$

then $\mathbf{F}$ is conservative.

We will prove Theorem 10 in the next section.

☐ 直觀上，單連通區域代表此區域沒有「洞」— 虧格爲零 (genus); 而且無法分成兩塊。

Finally, we will use “partial integration” to find the potential functions.

例題中的向量場處處都是一次微分仍連續，所以只要檢驗 $Q_x = P_y$ 就可確定它是否爲保守向量場。若要得到位勢函數，則用積分後再微分的方法求得。注意到若函數對 x 積分後，所有和 y 有關的函數 $g(y)$ 都會是「積分常數」。

Example 11 (page 1091)**.**

(a) If $\mathbf{F}(x,y) = (3+2xy)\,\mathbf{i} + (x^2-3y^2)\,\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

(b) Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the curve given by $\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j}$, and t from 0 to π.

Solution.

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對於空間中的向量場，同樣也是先利用函數的二次微分交換律而得到保守向量場的必要條件，它必須滿足三個方程式。

Example 12 (page 1095)**.** Show that if the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is conservative and P, Q, R have continuous first order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \text{and} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

Proof. Since $\mathbf{F} = \nabla f$, we have $P = f_x, Q = f_y$, and $R = f_z$. By Clairaut's Theorem, we know that $P_y = (f_x)_y = f_{xy} = f_{yx} = (f_y)_x = Q_x$, $P_z = (f_x)_z = f_{xz} = f_{zx} = (f_z)_x = R_x$, and $Q_z = (f_y)_z = f_{yz} = f_{zy} = (f_z)_y = R_y$. ☐

Example 13. If $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + (2xy + \mathrm{e}^{3z})\,\mathbf{j} + 3y\mathrm{e}^{3z}\,\mathbf{k}$, find a function f such that $\nabla f = \mathbf{F}$.

這個例題也是先練習如何確實找出位勢函數。

Solution.

Example 14 (page 1095)**.** Let $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} = \frac{-y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j}$.

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(a) Show that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

(b) Show that $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is *not* independent of path.

(c) Compute $\nabla\theta(x, y)$, where $\theta = \theta(x, y)$ is the polar angle function.

這個函數可以幫助我們仔細思考這一節所有理論的邏輯關係, 包括 $Q_x = Py$、保守向量場、積分與路徑選取無關、位勢函數的存在性, 必須自己把所有觀念重新整理一次。

Solution.

Appendix: Conservation of Energy, page 1093

0-xRNBTrdqY

至此我們可以利用現學到的微積分觀念證明物理課所學的能量守恆定律。

We will apply these ideas to a continuous force field $\mathbf{F}$ that moves an object along a path C given by $\mathbf{r}(t), a \le t \le b$, where $\mathbf{r}(a) = A$ is the initial point and $\mathbf{r}(b) = B$ is the terminal point of C.

According to Newton's Second Law of Motion, the force $\mathbf{F}(\mathbf{r}(t))$ at a point on C is related to the acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$ by the equation $\mathbf{F}(\mathbf{r}(t)) = m\,\mathbf{r}''(t)$, so the work done by the force on the object is

$$\begin{aligned} W &= \int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,\mathrm{d}t = \int_a^b m\,\mathbf{r}''(t)\cdot\mathbf{r}'(t)\,\mathrm{d}t \\ &= \frac{m}{2}\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}'(t)\cdot\mathbf{r}'(t))\,\mathrm{d}t = \frac{m}{2}\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{r}'(t)|^2\,\mathrm{d}t = \frac{m}{2}\Big[|\mathbf{r}'(t)|\Big]\Big|_a^b \\ &= \frac{m}{2}(|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2) = \frac{1}{2}m|\mathbf{v}(b)|^2 - \frac{1}{2}m|\mathbf{v}(a)|^2, \end{aligned} \tag{5}$$

where $\mathbf{v}(t) = \mathbf{r}'(t)$ is the velocity.

The quantity $\frac{1}{2}m|\mathbf{v}(t)|^2$, is called the *kinetic energy* (動能) of the object. Therefore we can rewrite Equation (5) as $W = K(B) - K(A)$, which says that the work done by the force field along C is equal to the change in kinetic energy at the endpoints of C.

Now let's further assume that $\mathbf{F}$ is a conservative force field; that is, we can write $\mathbf{F} = \nabla f$. In physics, the *potential energy* (位能) of an object at the point (x, y, z) is defined as $P(x, y, z) = -f(x, y, z)$, so we have $\mathbf{F} = \nabla f = -\nabla P$. Then we have

$$W = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = -\int_C \nabla P\cdot\mathrm{d}\mathbf{r} = -(P(\mathbf{r}(b)) - P(\mathbf{r}(a))) = P(A) - P(B).$$

Comparing this equation with $W = K(B) - K(A)$, we see that

$$P(A) + K(A) = P(B) + K(B),$$

which says that if an object moves from one point A to another point B under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant. This is called the *Law of Conservation of Energy* (能量守恆定律) and it is the reason the vector field is called *conservative*.

16.4 Green's Theorem, page 1096

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C.

XBngC_gleX8

在介紹格林定理之前，要先定義區域邊界的方向性，我們會說曲線以逆時針方向行走時爲正的定向，順時針方向行走時爲負的定向。

Definition 1 (page 1096)**.** We say a simple closed curve C is *positive orientation* (正的定向) if the curve is traverses counterclockwise.

If C is given by the vector function $\mathbf{r}(t), a \leq t \leq b$, then the region D is always on the left at the point $\mathbf{r}(t)$ traverses C.

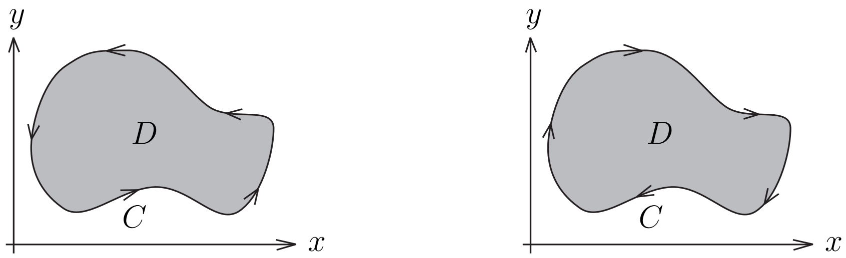


Figure 1: Positive orientation (left) and negative orientation (right).

Green's Theorem (page 1096)**.** *Let C be a positive oriented, piecewise smooth, simple closed curve in the plane and let D be the region bounded by C. If $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on an open region that contains D, then*

$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A.$$

關於格林定理，若與前一節觀念連結，則可想成封閉路徑積分是否爲零與向量場是否爲保守向量場的差有關。

Remark 2. Sometimes we use the following notations

$$\oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y, \qquad \oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y, \quad \text{or} \quad \int_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y$$

to indicate that the line integral is calculated in the positive orientation.

Example 3 (page 1098)**.** Evaluate $\int_C x^4\,\mathrm{d}x + xy\,\mathrm{d}y$, where C is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, form $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

JhdcsgJbJuA

利用格林定理，可以把第二類線積分的問題轉換爲區域內的二重積分。

Solution.

這個向量場若要直接計算線積分會很有難度， 因爲向量場的分量帶有積不出來的函數。取而代之的是， 利用格林定理，那些可怕的項微分後消失，所以可以順利進行重積分。

Example 4 (page 1098)**.** Evaluate $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$, where C is the circle $x^2 + y^2 = 9$.

Solution.

縱使在區域內部的向量場很複雜，但是在區域在邊界上的向量場都是零向量時，格林定理告知：區域內部對於 $Q_x - P_y$ 重積分的結果也是零。

Example 5 (page 1098)**.** If $P(x,y) = Q(x,y) = 0$ on a simple closed curve C, and $P(x,y), Q(x,y)$ satisfy the hypotheses of Green's Theorem, then

no matter what values P and Q assume in the region D.

LD8z1qffEGM

Example 6 (page 1099)**.** If we take $(P,Q) = (0,x)$, $(P,Q) = (-y,0)$, and $(P,Q) = (-\frac{1}{2}y, \frac{1}{2}x)$, then Green's Theorem gives

適當的設定 P, Q 讓 $Q_x - P_y = 1$， 則格林定理的右式會對到區域面積，而左式可以轉變爲線積分。

特別地，格林定理可以證明你以前可能學過的多邊形面積公式，只要把多邊形的頂點逆時針排列，把坐標依序標下（第一個點重新標註）之後，就有類似交叉相乘互減除以二的公式。

Example 7 (page 1102)**.**

(a) If C is the line segment connecting the point (x_1, y_1) to the point (x_2, y_2), then

$$\int_C -\frac{1}{2}y\,dx + \frac{1}{2}x\,dy = \frac{1}{2}(x_1y_2 - x_2y_1).$$

(b) If the vertices of a polygon, in counterclockwise order, are $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, then the area of the polygon is

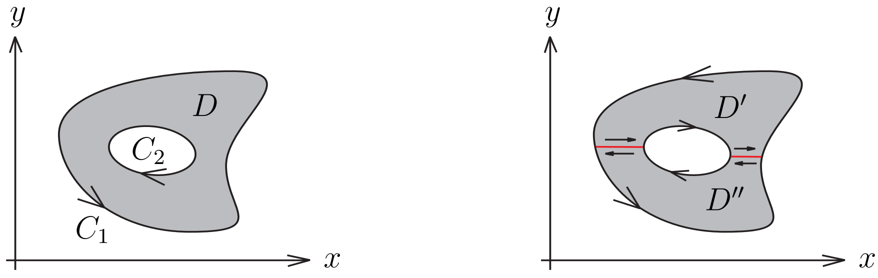
$$A = \frac{1}{2}((x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - y_{n-1}x_n) + (x_ny_1 - y_nx_1)).$$

Extended Versions of Green's Theorem, page 1099

Green's Theorem can be extended to apply to regions with holes (genus), that is, regions that are not simply connected.

kU5eA_o5XY0

區域的內部有洞的時候，在區域上 $Q_x - P_y$ 的重積分與邊界上的線積分必須重新討論，對外部邊界而言是取逆時針，內部邊界而言要取順時針，其結果才會一致。



(a) (b)

Figure 2: Region D is not simply connected.

See Figure 2 (a). Observe that the boundary C of the region D consists of two simple closed curves C_1 and C_2. We assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed. Thus the positive direction is *counterclockwise* for the outer curve C_1 but *clockwise* for the inner curve C_2.

If we divide D into two region D' and D'' by means of the lines shown in Figure 2 (b), then we applying Green's Theorem to each of D' and D'' to get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A = \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A$$
$$= \int_{\partial D'} P\,\mathrm{d}x + Q\,\mathrm{d}y + \int_{\partial D''} P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

Since the line integrals along the common boundary lines are in opposite directions, they cancel and we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A = \int_{C_1} P\,\mathrm{d}x + Q\,\mathrm{d}y + \int_{C_2} P\,\mathrm{d}x + Q\,\mathrm{d}y = \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

j_ZkqHvIEPY

這個向量場在 $(0,0)$ 處沒有定義，也就是說，向量場除了坐標原點以外是一次微分仍連續的向量場，向量場除了坐標原點外都滿足 $Q_x = P_y$，所以當一個封閉曲線內部不包含坐標原點時，線積分可用保守向量場的觀念處理，其值爲零。當卦閉曲線內部包含坐標原點時，積分值不爲零，用推廣版本的格定定理驗證包含坐標原點的封閉曲線的積分(逆時針)，與一個包含坐標原點的圓(逆時針) 的積分一樣，而圓形就可以確實地參數化並求值。

Example 8 (page 1100)**.** Let $\mathbf{F}(x,y) = \frac{-y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j}$.

(a) Show that $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every simple closed curve that does not encloses the origin.

(b) Show that $\int_C \mathbf{F}\cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

Solution.

Appendix, page 1097

Proof of Green's Theorem in which D is a simple region. It suffices to show that

$$\int_C P(x,y)\,\mathrm{d}x = -\iint_D \frac{\partial P}{\partial y}\,\mathrm{d}A \quad \text{and} \quad \int_C Q(x,y)\,\mathrm{d}y = \iint_D \frac{\partial Q}{\partial x}\,\mathrm{d}A.$$

BwISG5qgT6E

附錄是關於格林定理的證明。對於一個區域來說，進行切割之後讓每一小塊是滿足 type I 或是 type II 的區域，所以整個問題就化簡成分別討論 type I 或 type II 區域之下格林定理是否成立。更進一步地，格林定理的線積分與重積分的關係，可以看到和 P 有關的是一個等式，和 Q 有關的是另一個等式，格林定理是將這兩個結果合併。

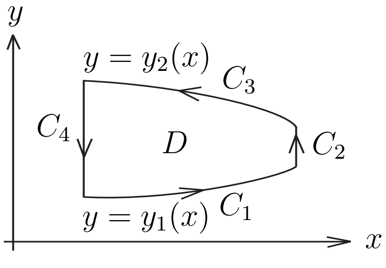


Figure 3: Simple Region D.

We express D as a type I region $D = \{(x,y) \,|\, a \le x \le b, y_1(x) \le y \le y_2(x)\}$, where $y_1(x)$ and $y_2(x)$ are continuous functions. By the Fundamental Theorem of Calculus, we have

$$-\iint_D \frac{\partial P}{\partial y}\,\mathrm{d}A = -\int_a^b \int_{y_1(x)}^{y_2(x)} \frac{\partial P}{\partial y}(x,y)\,\mathrm{d}y\,\mathrm{d}x = -\int_a^b \left(P(x,y_2(x)) - P(x,y_1(x))\right)\mathrm{d}x.$$

On the other hand, we know $C = C_1 \cup C_2 \cup C_3 \cup C_4$. On C_1, we write the vector function $\mathbf{r}_1(t) = t\,\mathbf{i} + y_1(t)\,\mathbf{j}$, and t from a to b. So

$$\int_{C_1} P(x,y)\,\mathrm{d}x = \int_a^b P(x,y_1(x))\,\mathrm{d}x.$$

On C_3, we use the vector function $\mathbf{r}_3(t) = t\,\mathbf{i} + y_2(t)\,\mathbf{j}$, t from b to a. Therefore

$$\int_{C_3} P(x,y)\,\mathrm{d}x = \int_b^a P(t,y_2(t))\,\mathrm{d}t = -\int_a^b P(x,y_2(x))\,\mathrm{d}x.$$

On C_2 or C_4, x is constant, so $\mathrm{d}x = 0$ and hence

$$\int_{C_2} P(x,y)\,\mathrm{d}x = 0 = \int_{C_4} P(x,y)\,\mathrm{d}x.$$

Hence

$$\begin{aligned}\int_C P(x,y)\,\mathrm{d}x &= \int_{C_1} P(x,y)\,\mathrm{d}x + \int_{C_2} P(x,y)\,\mathrm{d}x + \int_{C_3} P(x,y)\,\mathrm{d}x + \int_{C_4} P(x,y)\,\mathrm{d}x \\ &= \int_a^b P(x,y_1(x))\,\mathrm{d}x - \int_a^b P(x,y_2(x))\,\mathrm{d}x = -\iint_D \frac{\partial P}{\partial y}\,\mathrm{d}A.\end{aligned}$$

Equality $\int_C Q(x,y)\,\mathrm{d}y = \iint_D \frac{\partial Q}{\partial x}\,\mathrm{d}A$ can be proved similarly. □

Proof of Theorem 10 in section 16.3. If C is any simple closed path in D and R is the region that encloses, then Green's Theorem gives

hLomuqZIf84

這裡要補充證明前一節的定理：在單連通區域下的一次微分仍連續的向量場若滿足 $Q_x = P_y$，則它是保守向量場。

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A = \iint_R 0\,\mathrm{d}A = 0.$$

A curve that is not simple crossed itself at one or more points and can be broken up into a number of simple curve. We have shown that the line integral of $\mathbf{F}$ around these simple curves are all 0 and, adding these integrals, we see that $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ for any closed curve C. Therefore $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of path in D, and $\mathbf{F}$ is a conservative vector field. □

各位若之後有學到微分幾何的話，我們會用類似像外積的記號將面元賦予符號，透過這種符號可以將格林定理的敘述與微積分基本定理統一起來。

Remark 9. In differential geometry, we define the "wedge product" or "exterior operator" on vectors or differential forms. Given two differential forms $\mathrm{d}x, \mathrm{d}y$, their wedge product $\mathrm{d}x \wedge \mathrm{d}y$ means the positive oriented area element, so we have

$$\mathrm{d}A = \mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x, \quad \text{and} \quad \mathrm{d}x \wedge \mathrm{d}x = 0, \quad \text{and} \quad \mathrm{d}(\mathrm{d}x) = 0.$$

Green's Theorem can be regarded as the relationship between the integral, differential forms, and wedge product:

$$\begin{aligned}
&\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \mathrm{d}(P\,\mathrm{d}x + Q\,\mathrm{d}y) \\
&= \iint_D \left(\frac{\partial P}{\partial x}\,\mathrm{d}x + \frac{\partial P}{\partial y}\,\mathrm{d}y \right) \wedge \mathrm{d}x + P\,\mathrm{d}(\mathrm{d}x) + \left(\frac{\partial Q}{\partial x}\,\mathrm{d}x + \frac{\partial Q}{\partial y}\,\mathrm{d}y \right) \wedge \mathrm{d}y + Q\,\mathrm{d}(\mathrm{d}y) \\
&= \iint_D \frac{\partial P}{\partial y}\,\mathrm{d}y \wedge \mathrm{d}x + \iint_D \frac{\partial Q}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A.
\end{aligned}$$