Chapter 15 Multiple Integrals

15.1 Double Integrals over Rectangles, page 988

Review of the Definite Integral, page 988

Suppose that $f(x)$ is defined for $a \le x \le b$.

T9kxZXTvGGw

在介紹重積分之前，先複習單變數函數的定積分，透過分割、樣本點、取和、求極限的四個過程，希望這個極限值可以描述函數與 x-軸所圍出的區域面積，而這個定積分的值受到積分的順向逆向，還有函數圖形在 x-軸的上方或下方而允許有負值，所以定積分應進一步解釋爲帶有方向與符號的面積。

(1) Divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{n}$.

(2) In each subinterval $[x_{i-1}, x_i]$, choose a sample point $x_i^* \in [x_{i-1}, x_i]$.

(3) Define the Riemann sum (黎曼和) $= \sum\limits_{i=1}^{n} f(x_i^*)\Delta x$.

(4) Define the *definite integral* (定積分) of $f(x)$ from a to b by

$$\int_a^b f(x)\,\mathrm{d}x \overset{\text{def.}}{=} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

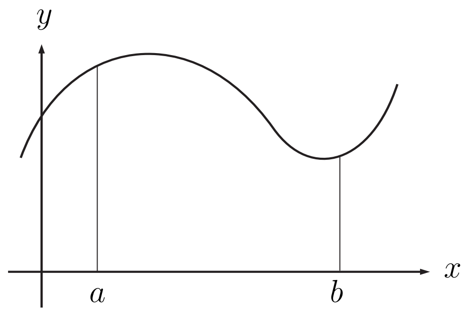

Figure 1: Definition of a definite integral.

☐ 幾何意義:「有向」面積; 函數在 x-軸的上方或下方; 積分範圍從左至右或從右至左。

Volumes and Double Integrals, page 988

現在要建立重積分的概念，也是試圖仿照分割、樣本點、取和、求極限的四個過程，希望這個極限值可以描述函數 $f(x, y)$ 與 xy-平面的區域間所圍出的實心物體體積。

Similarly, we consider a function $f(x, y)$ defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 | \, a \le x \le b, c \le y \le d\},$$

and we first suppose that $f(x, y) \ge 0$. The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f, that is

$$S = \{(x, y, z) \in \mathbb{R}^3 | \, 0 \le z \le f(x, y), (x, y) \in R\},$$

The goal is to find the volume of S.

(1) Divide the rectangle R into subrectangles:

- Divide $[a, b]$ into m subinterval $[x_{i-1}, x_i]$ with width $\Delta x = \frac{b-a}{m}$.
- Divide $[c, d]$ into n subinterval $[y_{i-1}, y_i]$ with width $\Delta y = \frac{d-c}{n}$.
- We form the subrectangles:
  $R_{ij} = [x_{i-1}, x_i] \times [y_{i-1}, y_i] = \{(x, y) | x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j\}$.
  Each R_{ij} with area $\Delta A = \Delta x \Delta y$.

(2) Choose a *sample point* (x_{ij}^*, y_{ij}^*) (樣本點) in each R_{ij}.

(3) We get an approximation to the total volume of S by *double Riemann sum* (二重黎曼和):

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A.$$

(4) Define the *volume* (體積) of the solid S by

$$V = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A.$$

重積分的定義, 在這裡要注意的是: 當我們使用 $\mathrm{d}A$ 這個符號時, 一律規定是正的面積, 所以只有 $f(x, y)$ 的正負才會影響重積分的正負。所以第十五章討論重積分對於區域的處理一律都取正量。第十六章討論向量微積分時才會特別研究區域的方向性與積分值的正負關係。

Definition 1 (page 990)**.** The *double integral* (重積分, 二重積分) of $f(x, y)$ over the rectangle R is

$$\iint_R f(x, y)\,\mathrm{d}A \overset{\text{def.}}{=} \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A$$

if this limit exists.

☐ 幾何意義:「有向」體積; 函數在 xy-平面的上方或下方。

☐ 寫成 $\mathrm{d}A$ 這個符號時, 總是代表「正的面元」。

☐ 使用記號 $\mathrm{d}x \wedge \mathrm{d}y$ (右手定則) 代表「有向面元」。($\mathrm{d}y \wedge \mathrm{d}x = -\mathrm{d}x \wedge \mathrm{d}y$)

Iterated Integrals, page 993

XZHTmMqg7eQ

Goal: Compute the double integrals by iterated integrals.
Recall that it is usually difficult to evaluate single integrals directly from the definition of an integral. The evaluation of double integrals from the definition is even more difficult. In this section, we will see how to express a double integral as an iterated integral, which can be evaluated by calculating two single integrals.

實際面來說, 我們很難直接用定義計算重積分, 取而代之的是, 我們想藉由單變數積分來了解重積分, 於是產生「二次積分」的概念。

Suppose that $f(x, y)$ is integrable on the rectangle $R = [a, b] \times [c, d]$.

Definition 2 (page 993)**.** Define the *partial integration of* $f(x, y)$ *with respect to* y, denoted by $\int_c^d f(x, y)\,\mathrm{d}y$ to mean that x is fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$.

After partial integration, $\int_c^d f(x, y)\,\mathrm{d}y$ depends on x, so we denote it by $A(x)$.

Definition 3 (page 993)**.** If we integrate $A(x) = \int_c^d f(x, y)\,\mathrm{d}y$ with respect to x from $x = a$ to $x = b$, we get the *iterated integral* (先對 y 後對 x 的二次積分):

將二變數函數一次對於一個變數積分的方式稱爲二次積分。因爲積分的先後關係，所以會有「先 y 後 x」與「先 x 後 y」兩種可能。

$$\int_a^b A(x)\,\mathrm{d}x = \int_a^b \left(\int_c^d f(x, y)\,\mathrm{d}y\right)\mathrm{d}x = \int_a^b\int_c^d f(x, y)\,\mathrm{d}y\,\mathrm{d}x$$

☐ 雖然有時候大括號會省略，但還是建議添加。

Similarly, the iterated integral (先對 x 後對 y 的二次積分)

$$\int_c^d\int_a^b f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left(\int_a^b f(x, y)\,\mathrm{d}x\right)\mathrm{d}y = \int_c^d B(y)\,\mathrm{d}y.$$

means that we first integrate with respect to x (fixed y) from $x = a$ to $x = b$ and then integrate the resulting function $B(y) = \int_a^b f(x, y)\,\mathrm{d}x$ from $y = c$ to $y = d$.

Fubini's Theorem (page 994)**.** *If* $f(x, y)$ *is continuous on the rectangles* $R = \{(x, y) | a \le x \le b, c \le y \le d\}$*, then*

對於連續函數，或是函數在有限條光滑曲線上不連續的且二次積分存在時，重積分與二次積分值一樣。高等微積分課程將深入討論這個定理，以及函數在更糟的時候重積分與二次積分不一致的情況。

$$\iint_R f(x, y)\,\mathrm{d}A = \int_a^b \left(\int_c^d f(x, y)\,\mathrm{d}y\right)\mathrm{d}x = \int_c^d \left(\int_a^b f(x, y)\,\mathrm{d}x\right)\mathrm{d}y.$$

In general, this is true if we assume that $f(x, y)$ *is bounded on* R*,* $f(x, y)$ *is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.*

xOzvZ-sWlOg

Example 4 (page 995)**.** Evaluate $\iint_R y\sin(xy)\,\mathrm{d}A$, where $R = [1, 2] \times [0, \pi]$.

Solution. If we first integrate with respect to x, we get

例題示範如何用二次積分計算重積分。這裡「先 x 後 y」與「先 y 後 x」兩種方法都確實呈現，此時會發現到有一種方式計算比較容易，而另一種方式會計算不易，所以什麼時候好算什麼時候不好算會是處理重積分時一個要研究的課題，甚至有些情況一種方式算得出來而另一種不行。

$$\iint_R y\sin(xy)\,\mathrm{d}A =$$

Solution 2. If we reverse the order of integration, we get

$$\iint_R y \sin(xy)\,\mathrm{d}A =$$

We use ____________________ and get

So

☐ 有時候只有一種方式「積得出來」, 所以積分「先後順序的轉換」要熟練並會巧妙變換。

roMh7_Iy5_4

例題示範用二次積分計算曲面與 xy-平面中的方形區域之間所圍出的實心物體體積。 這裡要開始學習如何將幾何問題確實轉換爲重積分再確實計算。

Example 5 (page 996)**.** Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the plane $x = 2$ and $y = 2$, and the three coordinate planes.

Solution.

zr0s4cGedmc

當函數是分離變數的形式時, 重積分的結果會是各別單變數函數積分值相乘。有時候一些實際問題中可以觀察函數的特性, 特別是周期函數可以用這個原理快速求值。

In the special case, where $f(x, y) = g(x)h(y)$ is the product of a function of x only and a function of y only, by Fubini's Theorem, we get

Example 6 (page 996)**.** If $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$, then

$$\iint_R \sin x \cos y\,\mathrm{d}A =$$

15.2 Double Integrals over General Regions, page 1001

Goal: We will learn how to integrate a function $f(x, y)$ over a bounded region D.

rsBTF6pBQ70

Define a new function $F(x, y)$ with a rectangular region $R \supseteq D$ by

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in D.} \end{cases} \tag{1}$$

這裡要建立的公式是對於非矩形區域下函數之重積分。先用一個較大的矩形把要研究的區域框住，再造一個定義於矩形區域的函數，這個函數在矩形內與區域外的地方定義成零，那麼原函數的重積分值與新函數的重積分值一樣。把定義域擴大成矩形的用意是想要用 Fubini 定理將問題轉換成二次積分，再對於積分區域改寫。

Figure 1: Double integral of f over D.

Definition 1 (page 1001)**.** If $F(x, y)$ is integrable over R, then we define the *double integral of* $f(x, y)$ *over* D (區域 D 上函數 $f(x, y)$ 的重積分) by

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_R F(x,y)\,\mathrm{d}A, \quad \text{where } F \text{ is given by (1).}$$

Definition 2 (page 1002)**.**

(1) A plane region D is said to be of *type I* if it lies between the graphs of two continuous functions of x, that is, $D = \{(x, y)|a \le x \le b, g_1(x) \le y \le g_2(x)\}$, where $g_1(x)$ and $g_2(x)$ are continuous on $[a, b]$.

(2) A plane region D is said to be of *type II* if it lies between the graphs of two continuous functions of y, that is, $D = \{(x, y)|h_1(y) \le x \le h_2(y), c \le y \le d\}$, where $h_1(y)$ and $h_2(y)$ are continuous on $[c, d]$.

這裡先觀察兩類比較特別的區域，依照區域邊界的屬性，若上下可視爲函數的圖形，稱爲 type I，若左右可視爲函數的圖形，稱爲 type II。

Figure 2: Type I and Type II region.

對於 type I 區域, 重積分可以轉變成先 y 後 x 的積分; 對於 type II 區域, 重積分可以轉變成先 x 後 y 的積分。注意到第一次積分的上下限會與後來要積分的變數有關, 也就是說, 上下限是與外層變數有關的函數。

Theorem 3 (page 1002–1003)**.**

(a) *If $f(x,y)$ is continuous on a type I region D, then*

$$\iint_D f(x,y)\,\mathrm{d}A = \int_a^b \left(\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x.$$

(b) *If $f(x,y)$ is continuous on a type II region D, then*

$$\iint_D f(x,y)\,\mathrm{d}A = \int_c^d \left(\int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,\mathrm{d}x\right)\mathrm{d}y.$$

Proof of (a). We choose a rectangle $R=[a,b]\times[c,d]$, where c and d are constants satisfy $c \le \min\limits_{x\in[a,b]} g_1(x)$ and $d \ge \max\limits_{x\in[a,b]} g_2(x)$. Let $F(x,y)$ be the function given by (1). By Fubini's Theorem, we have

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_R F(x,y)\,\mathrm{d}A = \int_a^b \left(\int_c^d F(x,y)\,\mathrm{d}y\right)\mathrm{d}x.$$

For fixed $x\in[a,b]$, since $F(x,y)=0$ if $y<g_1(x)$ or $y>g_2(x)$, the lower limit can be replaced by $g_1(x)$, and the upper limit can replaced by $g_2(x)$. Therefore,

$$\iint_D f(x,y)\,\mathrm{d}A = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} F(x,y)\,\mathrm{d}y\right)\mathrm{d}x = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x$$

because $F(x,y)=f(x,y)$ when $g_1(x)\le y\le g_2(y)$. □

- ☐ 非矩形區域上函數的積分, 上下限範圍改成「函數」。
- ☐ 若是先對 y 積分, 上、下限當然是 y 的上下限, 所以是 $y=g_2(x)$ 與 $y=g_1(x)$。
- ☐ 學會畫圖 (定義域); 熟練區域變換。

r0dx3cDKjtY

例題示範當區域是 type I 時, 如何將重積分轉變成先 y 後 x 的二次積分。

Example 4 (page 1003)**.** Evaluate $\iint_D (x+2y)\,\mathrm{d}A$, where D is the region bounded by the parabolas $y=2x^2$ and $y=x^2+1$.

Example 5. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = x$ and the parabola $y = x^2$.

z3YD9DoP-a0

這個例題積分的區域既是 type I 也是 type II, 所以兩種方式都確實呈現。計算重積分時, 要先把區域確實標出, 解讀它是哪一類形的積分, 當區域邊界是上下爲函數時 (type I), 由下往上畫一條與 y 軸平行的線, 則線條第一次進入區域的邊界函數就是積分的下限, 最後從區域穿出的函數會是積分的上限。積分完之後, 將區域投影到 x-軸, 則左端點爲積分下限, 右端點爲積分上限。

Solution.

Solution 2.

Example 6. Evaluate $\iint_D xy \, \mathrm{d}A$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = x + 1$.

tQAvwDWfOAY

當區域邊界是左右爲函數時 (type II), 由左往右畫一條與 x 軸平行的線, 則線條第一次進入區域的邊界函數就是積分的下限, 最後從區域穿出的函數會是積分的上限。積分完之後, 將區域投影到 y-軸, 則下端點爲積分下限, 上端點爲積分上限。

Solution.

bj6YIiZV5hU

兩個圓柱垂直交集出的實心物體體積，可用單變數函數處理，現在用重積分的方法再次理解並計算。對於這個問題，重積分的計算方法要先徹底了解，之後我們還要挑戰計算三個圓柱互相垂直貫穿下的實心物體體積。

Example 7 (page 449, 1008)**.** Find the volume common to two circular cylinders, each with radius r, if the axes of the cylinders intersect at right angles.

Solution.

Solution 2.

IFv8aqDayJY

函數 $\sin(y^2)$ 對 y 積分是「積不出來」的，所以不可能直接計算得到結果，然而這個二次積分的值仍然算得出來，必須要將積分轉變成先 x 後 y 的二次積分。於是要先從原先二次積分的上下限解讀出積分區域，再重新用 type II 的方式重新改寫二次積分，再將結果算出。

Example 8 (page 1006)**.** Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\,\mathrm{d}y\,\mathrm{d}x$.

Solution.

Properties of Double Integrals, page 1006

We assume that all of the integrals exist.

(a) $\iint_D (f(x,y)+g(x,y))\,\mathrm{d}A = \iint_D f(x,y)\,\mathrm{d}A + \iint_D g(x,y)\,\mathrm{d}A.$

(b) $\iint_D cf(x,y)\,\mathrm{d}A = c\iint_D f(x,y)\,\mathrm{d}A.$

(c) If $f(x,y) \geq g(x,y)$ for all $(x,y) \in D$, then $\iint_D f(x,y)\,\mathrm{d}A \geq \iint_D g(x,y)\,\mathrm{d}A.$

(d) If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except on boundaries, then

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D_1} f(x,y)\,\mathrm{d}A + \iint_{D_2} f(x,y)\,\mathrm{d}A.$$

This property can be used to evaluate double integrals over regions D that are neither type I nor type II but can be expressed as a union of regions of type I or type II.

Figure 3: D is neither type I nor type II. D_1 is type I, D_2 is type II.

(e) $\iint_D 1\,\mathrm{d}A = \mathrm{Area}(D).$

(f) If $m \leq f(x,y) \leq M$ for all (x,y) in D, then

$$m\,\mathrm{Area}(D) \leq \iint_D f(x,y)\,\mathrm{d}A \leq M\,\mathrm{Area}(D).$$

jJRIgsDeHuE

這裡重新統整重積分與幾何圖形之間的對應。就 (a) 與 (b)，以線性代數的語言來說，積分是函數空間中的一個線性變換。對於 (c)，函數有大小關係導致重積分也有大小關係，這與實心物體的體積大小比較觀點一致。而 (d) 告知重積分可以對於區域分解，也和實體體積的分解一致。這一節學到的是 type I 與 type II 區域上的重積分處理法，對於更複雜的區域，我們就要設法把區域分解，就可以各別處理再相加。對於 (e), 雖然我們原先對於重積分的理解是在算體積，但是當函數取成 1 的時候, 計算出來的結果也與區域面積一致（忘掉單位，只看數值）。最後 (f) 的結果是對於重積分的值有一個最粗略的估計。

15.3 Double Integrals in Polar Coordinates, page 1010

8W7oAWCQjNQ

Goal: We want to evaluate a double integral $\iint_R f(x,y)\,\mathrm{d}A$, where R is easily described using polar coordinates.

Recall that relations between Cartesian coordinates and polar coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x}. \end{cases}$$

這單元的學習目標是當區域用極坐標表示時，定義在區域上的函數該如何用極坐標的變數進行二重積分或二次積分。

Definition 1 (page 1010)**.** Define the *polar rectangle* (極坐標的範圍)

$$R = \{(r,\theta) |\, a \le r \le b, \alpha \le \theta \le \beta\}.$$

這裡也是從頭開始建構，先對於極坐標變數的上下範圍都是常數討論起。這時，極坐標的範圍對應到平面上的圖形會是大扇形扣掉小扇形的區域。再依照分割、樣本點、取和、求極限的過程得到重積分公式。

Here we use the definition of double integral to find the formula of double integrals in polar rectangles.

(1) Divide the polar rectangle into small polar rectangles:
   - Dividing $[a,b]$ into m subinterval $[r_{i-1}, r_i]$ with width $\Delta r = \frac{b-a}{m}$.
   - Dividing $[\alpha,\beta]$ into n subinterval $[\theta_{i-1}, \theta_i]$ with width $\Delta\theta = \frac{\beta-\alpha}{n}$.
   - We get small polar rectangles: $R_{ij} = \{(r,\theta) | r_{i-1} \le r \le r_i, \theta_{j-1} \le \theta \le \theta_j\}$.

這裡要注意的是，重積分原本是要求任意選取樣本點之下極限都要存在，只是這裡的重點是要推得極坐標二次積分的公式，所以這時對於 r_i^* 這個樣本點來說，選取中點，而 θ_j^* 來說選任意的角度都可以得到這個公式。也就是說，以數學的論述嚴謹度來說，在這些討論之前必須先證明不管怎麼選樣本點之下二重極限都要一樣，然後才特別選擇區域中點爲樣本點以得到積分公式。

(2) Choose the "center" of the polar subrectangle (r_i^*, θ_j^*) as the *sample point* in each R_{ij}, where

$$r_i^* = \frac{1}{2}(r_{i-1} + r_i), \qquad \theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j).$$

We compute the area of R_{ij}:

$$\Delta A_{ij} = \frac{1}{2} r_i^2 \Delta\theta - \frac{1}{2} r_{i-1}^2 \Delta\theta = \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta = r_i^* \Delta r \Delta\theta.$$

(3) We get the double Riemann sum in polar rectangles:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\Delta A_i = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*) r_i^* \Delta r \Delta\theta.$$

(4) When $m, n \to \infty$, we get

$$\begin{aligned} \iint_R f(x,y)\,\mathrm{d}A &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*) r_i^* \Delta r \Delta\theta \\ &= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r \mathrm{d}r\mathrm{d}\theta. \end{aligned}$$

Change to Polar Coordinates in a Double Integral (page 1012)**.** *If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then*

$$\iint_R f(x,y)\,\mathrm{d}A = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta.$$

這裡列出重積分在極坐標下的積分公式，重點在積分面元 $r\,\mathrm{d}r\,\mathrm{d}\theta$ 應確實理解。

- ☐ 極坐標的面元是 $r\,\mathrm{d}r\,\mathrm{d}\theta$; 幾何上可想成微小區域面積 $\mathrm{d}r\,r\mathrm{d}\theta$: 徑向長乘上圓弧長。
- ☐ $r\,\mathrm{d}r\,\mathrm{d}\theta$ 前面的 r 稱爲 Jacobian: $\frac{\partial(x,y)}{\partial(r,\theta)} = \left|\det\left(\begin{bmatrix} x_r & x_\theta \\ y_r & y_\theta \end{bmatrix}\right)\right|$。

Example 2 (page 1012)**.** Evaluate $\iint_R (3x + 4y^2)\,\mathrm{d}A$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

wEBoecONbUI

例題示範如何用極坐標改寫區域邊界並處理積分。

這裡有一件事情要注意: 計算當中有用到函數是分離變數型，所以可以改成各自積分後再相乘，但是這個性質必須建立在積分上下限都是常數的時候才可以使用。這一節的後面將討論更複雜的區域，到時候二次積分將導致積分的上下限不再是常數，就不能用這個性質了。

Solution.

Example 3 (page 1012)**.** Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

_-2oeGact98

計算實心物體的體積時，當函數投影到 xy-平面上的區域比較像是圓形或扇形時，可以先嘗試使用極坐標處理積分。這裡必須強調的是: 坐標只是表現區域的一種方法，只是長得像圓形或扇形的區域用極坐標表示通常會比較簡單，但是並沒有說這類的區域只能用極坐標處理積分。換言之，你也可以試著用直角坐標積分，或是前一節的題目試著用極坐標積分處理。

Solution.

XFIXBRVJ8BI

Theorem 4 (page 1013)**.** *If $f(x,y)$ is continuous on a polar region $D = \{(r,\theta) | \, h_1(\theta) \le r \le h_2(\theta), \alpha \le \theta \le \beta\}$, then*

當區域邊界可由極坐標方程改寫時，重積分公式的積分上下限應做調整。

$$\iint_D f(x,y)\,\mathrm{d}A = \int_\alpha^\beta \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta.$$

☐ 先對 r 方向積分, 上限爲 $r = h_2(\theta)$, 下限爲 $r = h_1(\theta)$, 皆爲 θ 的函數。

四葉玫瑰線所圍之區域面積可用單變數積分處理，這裡用重積分的方式處理同樣的問題。當函數設成 1，則重積分的值與區域面積大小一致。積分的時候，先從極點畫出一條射線，射線第一次進入區域的地方就是積分下限，最後穿出區域的線條就是積分上限。再看張角的變化最大與最小即爲積分的上下限。

Example 5 (page 1013)**.** Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Solution.

h1I4M2wlds4

Example 6 (page 1014)**.** Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

Solution.

極坐標的設定，不見得極點一定要和直角坐標的中心一致，可以根據問題而調整適合的坐標系。這個例子兩種方法都可解，但是會有一些題目，改用平移過後的極坐標反而重積分積不出來，所以原始的極坐標系下將曲線轉成極坐標方程式仍有學習的必要。

Appendix

附錄所寫，是想用微分幾何的語言解釋面積元素，它可看成是坐標曲線的切向量所張出的平行四邊形面積，若用 $\wedge$ 這個記號，它是帶有方向性的。這裡只是列出來供各位參考，微積分課程中還是先以無向面積 $\mathrm{d}A$ 的方式討論。

The area form (or volume form) has the following structure: Since

$$\begin{cases} \mathrm{d}x = \mathrm{d}r\cos\theta - r\sin\theta\,\mathrm{d}\theta \\ \mathrm{d}y = \mathrm{d}r\sin\theta + r\cos\theta\,\mathrm{d}\theta \end{cases} \Rightarrow \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} \mathrm{d}r \\ \mathrm{d}\theta \end{bmatrix}.$$

We compute the determinant of the matrix, called *Jacobian*:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Thus the area form is $\mathrm{d}x \wedge \mathrm{d}y = r\,\mathrm{d}r \wedge \mathrm{d}\theta$.

15.4 Applications of Double Integrals, page 1016

In this section we explore physical applications such as computing mass, electric charge, center of mass, and moment of inertia. We will apply double integrals to probability density functions of two random variables as well.

kwIAnSYR04g

這一些利用一些物理量來說明重積分的應用。考慮一個面板，若知道面板的密度函數，將密度函數重積分則得質量。若是考慮電荷密度，則將電荷密度積分後得到總電荷量。

Density and Mass, page 1016

Consider a lamina with variable density. Suppose the lamina occupies a region D of the xy-plane and its *density* (in units of mass per unit area) at a point (x, y) in D is given by $\rho(x, y)$, where ρ is a continuous function on D, then the total mass is

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D \rho(x, y)\,\mathrm{d}A.$$

If an electric charge is distributed over a region D and the charge density (in units of charge per unit area) is given by $\sigma(x, y)$ in D, then the total charge Q is

$$Q = \iint_D \sigma(x, y)\,\mathrm{d}A.$$

Example 1 (page 1017)**.** Charge is distributed over the region D bounded by $x = 1, y = 1$, and $x + y = 1$. The charge density at (x, y) is $\sigma(x, y) = xy$, measured in coulombs per square meter (C/m^2). Find the total charge.

例題示範電荷密度不均的區域如何計算總電荷量。

Solution.

Moments and Centers of Mass, page 1017

Suppose that the lamina occupies a region D and has density function $\rho(x, y)$. The *moment* of the entire lamina *about the y-axis* is

pKxqglgoj30

$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D x\rho(x, y)\,\mathrm{d}A.$$

重積分在物理上的另一個應用是得到不規則物體的質心，希望這個面板的一些物理效應，與一個質量爲 m 的質點放在質心位置的物理效應一樣。

Similarly, the *moment about the x-axis* is

$$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D y\rho(x, y)\,\mathrm{d}A.$$

The *center of mass* $(\bar{x}, \bar{y})$ of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x\rho(x, y)\,\mathrm{d}A \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y\rho(x, y)\,\mathrm{d}A,$$

where the mass m is given by $m = \iint_D \rho(x, y)\,\mathrm{d}A$.

Example 2 (page 1018)**.** The density at any point on a semicircular lamina with radius R is proportional to the distance from the center of the circle. Find the center of mass of the lamina.

Solution.

Moment of Inertia, page 1019

eox0t-aR75A

轉動慣量的概念是一個物體若要對著一個軸旋轉時，用量化的方式描述轉動的困難度。

The *moment of inertia* (also called the *second moment*) of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis. The *moment of inertia* of the lamina *about the x-axis* is

$$I_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D y^2 \rho(x, y)\,\mathrm{d}A.$$

Similarly the *moment of inertia about the y-axis* is

$$I_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D x^2 \rho(x, y)\,\mathrm{d}A.$$

It is also of interest to consider the *moment of inertia about the origin*, also called the *polar moment of inertia*:

$$I_0 = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left((x_{ij}^*)^2 + (y_{ij}^*)^2\right) \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D (x^2 + y^2)\rho(x, y)\,\mathrm{d}A.$$

Note that $I_0 = I_x + I_y$. The *radius of gyration* (迴轉半徑) of a lamina about an axis is the number R such that $mR^2 = I$, where m is the mass of the lamina and I is the moment of inertia about the given axis. In particular, the radius of gyration $\bar{\bar{y}}$ with respect to the x-axis and the radius of gyration $\bar{\bar{x}}$ with respect to the y-axis are given by the equation $m\bar{\bar{y}}^2 = I_x$ and $m\bar{\bar{x}}^2 = I_y$.

Example 3 (page 1020–1021)**.**

(a) Find the moments of inertia I_x, I_y, and I_0 of a homogeneous disk D with density $\rho(x, y) = \rho$, center the origin, and radius R.

(b) Find the radius of gyration about the x-axis of the disk D.

Solution.

15.5 Surface Area, page 1026

jgxC76VEwQk

在幾何上，我們可以用重積分計算曲面的表面積。這一節要先討論的是若曲面可以表示成函數的圖形時表面積的公式。更一般的曲面表面積公式會在第十六章介紹。先在定義域上做矩形區域的切割，每一塊小矩形內部對上去的函數圖形，會一階近似於曲面在分割點的切平面在區域內部對應到的平行四邊形。將每個平形四邊形面積取和再求極限，即得曲面面積公式。

Goal: Find the formula of the surface area of the graph of $f(x, y)$.

Let S be a surface with equation $z = f(x, y)$, where $f(x, y)$ has continuous partial derivatives. We assume that $f(x, y) \geq 0$ and the domain D of f is a rectangle. The idea is to approximate the surface area by the "tangent plane areas."

(1) Divide D into small rectangles R_{ij} with area $\Delta A = \Delta x \Delta y$.

(2) If we choose (x_i, y_i), the corner of R_{ij} closest to the origin, as a sample point, then the tangent plane to S at $P_{ij} = (x_i, y_j, f(x_i, y_j))$ is an approximation to S near P_{ij}. The area ΔT_{ij} of the part of this tangent plane that lies directly above R_{ij} is an approximation to the area ΔS_{ij} of the part of S that lies directly above R_{ij}.

$$\begin{aligned}\Delta T_{ij} &= |\mathbf{u}_i \times \mathbf{v}_j| = |(\Delta x\,\mathbf{i} + f_x(x_i, y_i)\Delta x\,\mathbf{k}) \times (\Delta y\,\mathbf{j} + f_y(x_i, y_j)\Delta y\,\mathbf{k})| \\ &= |-f_x(x_i, y_i)\Delta x \Delta y\,\mathbf{i} - f_y(x_i, y_i)\Delta x \Delta y\,\mathbf{j} + \Delta x \Delta y\,\mathbf{k}| \\ &= \sqrt{1 + (f_x(x_i, y_i))^2 + (f_y(x_i, y_i))^2}\,\Delta A.\end{aligned}$$

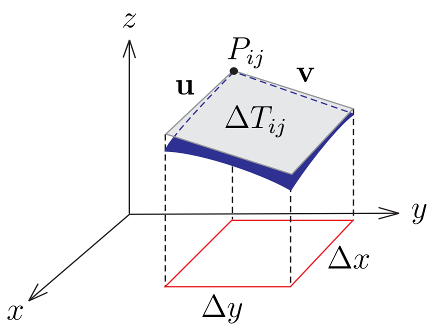

Figure 1: The area of a parallelogram $\Delta T_{ij} = |\mathbf{u}_i \times \mathbf{v}_j|$.

(3) The sum $\sum_{i=1}^{m}\sum_{j=1}^{n} \Delta T_{ij}$ is an approximation to the total area of S.

(4) We define the *surface area* (曲面面積) of S to be

$$\text{Area}(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \sqrt{1 + (f_x(x_i, y_i))^2 + (f_y(x_i, y_i))^2}\,\Delta A.$$

曲面表面積可想成是加權（與底部面積相比之下面積增加的倍率）後的重積分。也應該再與直角三角形的底邊與斜邊的畢氏定理關係與維度的關係聯想。

Theorem 1 (page 1027). *The area of the surface with equation $z = f(x, y)$, $(x, y) \in D$, where f_x and f_y are continuous, is*

$$\text{Area}(S) = \iint_D \sqrt{1 + (f_x(x, y))^2 + (f_y(x, y))^2}\,\mathrm{d}A.$$

☐ 類比於平面曲線 $(x, y = f(x)), a \leq x \leq b$ 的弧長公式: $L = \int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x$.

Example 2 (page 1027)**.** Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T in the xy-plane with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.

Y47Mynz-TFY

曲面表面積的具體示範。 這個例題雖然積分區域既是 type I 也是 type II, 概念上兩種二次積分都可以直接列式, 但是能否「積得出來」, 會因爲被積分的函數的屬性, 可能只有一種方式才算得出來。

Solution.

Example 3 (page 1028)**.** Find the area of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies within the cylinder $x^2 + y^2 = 2x$.

laN_Lcp7yuA

球面落在圓柱內部的表面積計算, 因爲積分的區域是圓盤, 所以在列式完之後, 將重積分改用極坐標的方式處理。

Solution.

15.6 Triple Integrals, page 1029

1U7HSigYuxU

這一節要開始處理三變數函數的重積分。也是從頭以立方體的區域進行分割、樣本點、取和、求極限定義三重積分, 同樣透過 Fubini 定理將三重積分與三次積分聯繫。實際計算時也是以如何轉換成三次積分的方式再逐次積分。

Goal: Define and compute triple integrals of $f(x,y,z)$ over a bounded region.

We first deal with the case where $f(x,y,z)$ is defined on a rectangular box:

$$B = \{(x,y,z) \,|\, a \le x \le b, c \le y \le d, r \le z \le s\}.$$

(1) Divide the box B into lmn sub-boxes: $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$. Each B_{ijk} has area $\Delta V = \Delta x \Delta y \Delta z$.

(2) Choose a *sample point* $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in each B_{ijk}.

(3) We get the *triple Riemann sum* (三重黎曼和): $\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V$.

(4) **Definition 1** (page 1030)**.** The *triple integral* (三重積分) of f over B is

$$\iiint_B f(x,y,z)\,\mathrm{d}V = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V$$

if this limit exists.

☐ 三變數函數在三度空間中無法再畫成函數的圖形 (必須放到四度空間, 但不好想像)。

☐ 三變數函數積分的一個理解方式如: 想成超厚牛排上熱量 (卡路里) 的總和。

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

Fubini's Theorem for Triple Integrals (page 1030)**.** *If f is continuous on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$, then*

$$\iiint_B f(x,y,z)\,\mathrm{d}V = \int_r^s \int_c^d \int_a^b f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Example 2 (page 1030)**.** Evaluate the triple integral $\iiint_B xyz^2\,\mathrm{d}V$, where $B = [0,1] \times [-1,2] \times [0,3]$.

Solution. Direct computation gives

$$\iiint_B xyz^2\,\mathrm{d}V = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\,\mathrm{d}y\,\mathrm{d}z = \int_0^3 \frac{3z^2}{4}\,\mathrm{d}z = \frac{27}{4}.$$

Now we define the *triple integral over a general bounded region E* (一般有界區域的三重積分) in three dimensional space (a solid). We enclosed E in a box B, and then define $F(x,y,z)$ on B that it agrees with $f(x,y,z)$ on E but is 0 in B that outside E. By definition,

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iiint_B F(x,y,z)\,\mathrm{d}V.$$

This integral exists if $f(x,y,z)$ is continuous and the boundary of E is "reasonably smooth."

JD1YokPHm70

爲了要處理定義域是不規則實體的三重積分，也是要設法轉變成三次積分再計算，此時也必須先對實體進行分類，根據上下、左右、前後是否可以表示成函數的圖形，而定義出 type z, type y, type x 區域。

Definition 3 (page 1031).

(a) Region E is called *type* x if $E = \{(x, y, z)|(x, y) \in D, x_1(y, z) \le x \le x_2(y, z)\}$, where D is the projection of E onto the yz-plane.

(b) Region E is called *type* y if $E = \{(x, y, z)|(x, y) \in D, y_1(x, z) \le y \le y_2(x, z)\}$, where D is the projection of E onto the xz-plane.

(c) Region E is called *type* z if $E = \{(x, y, z)|(x, y) \in D, z_1(x, y) \le z \le z_2(x, y)\}$, where D is the projection of E onto the xy-plane.

Now we will change triple integrals to iterated integrals.

■ If E is *type* z and D is *type* I, then

$$\iiint_E f(x, y, z)\,\mathrm{d}V = \int_{x=a}^{x=b} \int_{y=y_1(x)}^{y=y_2(x)} \int_{z=z_1(x,y)}^{z=z_2(x,y)} f(x, y, z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

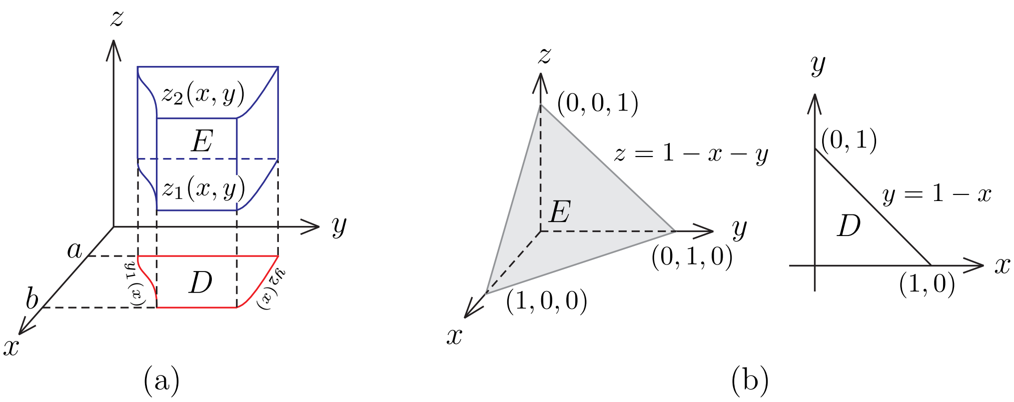

Figure 1: (a) Triple Integrals. (b) Solid E in **Example 4**.

Example 4 (page 1032). Evaluate $\iiint_E z\,\mathrm{d}V$, where E is the solid tetrahedron bounded by the four planes $x = 0, y = 0, z = 0$, and $x + y + z = 1$.

當積分的實體上下可以看成函數的圖形時，就可以優先對 z 進行積分，先畫出一條由下到上與 z 軸平行的直線，直線穿入實體的部分寫成積分的下限，直線穿出實體的表面寫成積分的上限。積分完畢之後將實體投影，就得到在 xy-平面上的區域，此時就回到 15.2 處理二重積分時的問題。

Solution.

Lep-sT10aOw

這個題目的拋物面開口對著 y-軸正項, 而另一個平面也是 y 爲常數, 所以這個三重積分可以優先對 y 進行積分。從左到右畫一條與 y-軸平行的線, 直線穿入實體的表面寫成積分的下限, 穿出的即爲積分的上限。將實體投影到 xz-平面之後, 發現區域是圓形, 所以可以試著用極坐標的方式處理二重積分。

Example 5 (page 1032)**.** Evaluate $\iiint_E \sqrt{x^2+z^2}\,\mathrm{d}V$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Solution.

IeihLQsrD14

個人認爲這是第十五章的最大難題。先從最外面兩層的積分上下限解讀區域, 再用最內層的積分上下限得到實體。再根據問題的要求重新改寫指定的三次積分。

Example 6 (page 1034)**.** Rewrite the iterated integral $\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$ in a different order, integrating first with respect to x, then z, and then y.

Solution.

Applications of Triple Integrals, page 1034

6pQ_-FMZEpA

當函數設定成 1, 則三重積分的值可代表實體體積。各位如果以前有背過四面體的體積公式, 可以和這個例子計算的結果對照。

If $f(x,y,z) \equiv 1$, then the triple integral represents the volume: $V(E) = \displaystyle\iiint_E 1\,\mathrm{d}V$.

Example 7 (page 1035)**.** Use a triple integral to find the volume of the tetrahedron T bounded by the plane $x + 2y + z = 2, x = 2y, x = 0$, and $z = 0$.

Solution.

Definition 8 (page 1035–1036)**.**

(a) If the density function of a solid object that occupies the region E is $\rho(x, y, z)$ (in units of mass per unit volume), then its *mass* (質量) is

$$m = \iiint_E \rho(x, y, z)\,\mathrm{d}V.$$

(b) The *moments* (矩) of region E about the three coordinate planes are

$$M_{yz} = \iiint_E x\rho(x, y, z)\,\mathrm{d}V, \qquad M_{xz} = \iiint_E y\rho(x, y, z)\,\mathrm{d}V,$$
$$M_{xy} = \iiint_E z\rho(x, y, z)\,\mathrm{d}V.$$

(c) The *center of mass* (質心) is located at the point $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m}.$$

(d) The *moments of inertia* (轉動慣量) about the three coordinate axes are

$$I_x = \iiint_E (y^2 + z^2)\rho(x, y, z)\,\mathrm{d}V, \quad I_y = \iiint_E (x^2 + z^2)\rho(x, y, z)\,\mathrm{d}V,$$
$$I_z = \iiint_E (x^2 + y^2)\rho(x, y, z)\,\mathrm{d}V.$$

Vx7ka3zyhQ4

三重積分也可以應用在物理上，也有質量、質心、轉動慣量等觀念。

Definition 9 (page 1036)**.** The total *electric charge* on a solid object occupying a region E and having charge density $\sigma(x, y, z)$ is

$$Q = \iiint_E \sigma(x, y, z)\,\mathrm{d}V.$$

Example 10 (page 1036)**.** Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$, and the planes $x = z, z = 0$, and $x = 1$.

Solution.

Appendix

附錄只是列舉其它五種三次積分的表達公式，關於三重積分轉換成三次積分，應多練習題目，每寫一次就推理一次， 練習足夠的題目就可以掌握要領。

- If E is *type* z and D is *type II*, then

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_{y=c}^{y=d}\int_{x=x_1(y)}^{x=x_2(y)}\int_{z=z_1(x,y)}^{z=z_2(x,y)} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y.$$

- If E is *type* y and D is *type I* $(z = z_i(x))$, then

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_{x=a}^{x=b}\int_{z=z_1(x)}^{z=z_2(x)}\int_{y=y_1(x,z)}^{y=t_2(x,z)} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x.$$

- If E is *type* y and D is *type II* $(x = x_i(z))$, then

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_{z=r}^{z=s}\int_{x=x_1(z)}^{x=x_2(z)}\int_{y=y_1(x,z)}^{y=y_2(x,z)} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z.$$

- If E is *type* x and D is *type I* $(z = z_i(y))$, then

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_{y=c}^{y=d}\int_{z=z_1(y)}^{z=z_2(y)}\int_{x=x_1(y,z)}^{x=x_2(y,z)} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y.$$

- If E is *type* x and D is *type II* $(y = y_i(z))$, then

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_{z=r}^{z=s}\int_{y=y_1(z)}^{y=y_2(z)}\int_{x=x_1(y,z)}^{x=x_2(y,z)} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

15.7 Triple Integrals in Cylindrical Coordinates, page 1040

Goal: Compute triple integrals in cylindrical coordinates.

7PmeKYSL2mE

Cylindrical Coordinates, page 1040

In the *cylindrical coordinate system* (柱坐標系), a point P in three-dimensional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the distance from P to the xy-plane.

關於三重積分的處理還有幾種常見的坐標系統，這裡要介紹的是柱坐標。柱坐標基本上是極坐標與直角坐標的綜合體，對 xy-平面的點而言，改用極坐標 (r, θ) 的方式呈現，而 z 的方向保持不動。

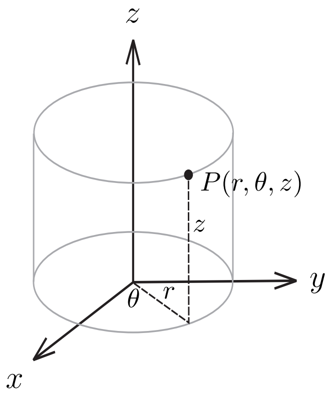

Figure 1: Cylindrical coordinate system.

- Relation from cylindrical to rectangular: $x = r\cos\theta, y = r\sin\theta, z = z$.
- Relation from rectangular to cylindrical: $r^2 = x^2 + y^2, \tan\theta = \frac{y}{x}, z = z$.

認識新的坐標系統時，必須認識各種坐標系之間的轉換公式與對應關係。

Example 1 (page 1041)**.**

(a) Find rectangular coordinates of the point with cylindrical coordinates $(2, \frac{2}{3}\pi, 1)$.

(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

Solution.

(a) Since $x =$ ____________________, $y =$ ________________, $z =$ ___. the point is ____________ in rectangular coordinates.

(b) Since $r =$ _______________, $\tan\theta =$ __________________, $z =$ ___, one of cylindrical coordinates is _____________, and another is ______________. As with polar coordinates, there are infinitely many choices.

Example 2 (page 1041)**.** Describe the surface whose equation in cylindrical coordinates is $z = r$.

Solution. Since $z^2 = r^2 = x^2 + y^2$, it is a ____________ whose axis is the z-axis.

第一次看到柱坐標下 $z = r$ 的方程式時可能無法一時意會它是錐面，但錐面的特性就是 z 值與 r 成正比的點所成之集合。

Evaluating Triple Integrals with Cylindrical Coordinates, page 1042

w1T7kfb7X08

使用柱坐標改寫三重積分時，對於 z 的部份，實體的上下邊界可以變成函數的圖形。而實體投影到 xy 平面後，區域的邊界改用極坐標方程式呈現，然後轉變成積分的上下限。

Suppose that E is a *type z* region whose projection D onto the xy-plane is conveniently described in polar coordinates. Suppose that $f(x, y, z)$ is continuous and

$$E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\},$$

where D is given in polar coordinates by $D = \{(r, \theta) | \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$. We get *the formula for triple integration in cylindrical coordinates*:

$$\iiint_E f(x, y, z)\, \mathrm{d}V = \int_\alpha^\beta \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=u_1(r\cos\theta, r\sin\theta)}^{z=u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r\, \mathrm{d}z\, \mathrm{d}r\, \mathrm{d}\theta.$$

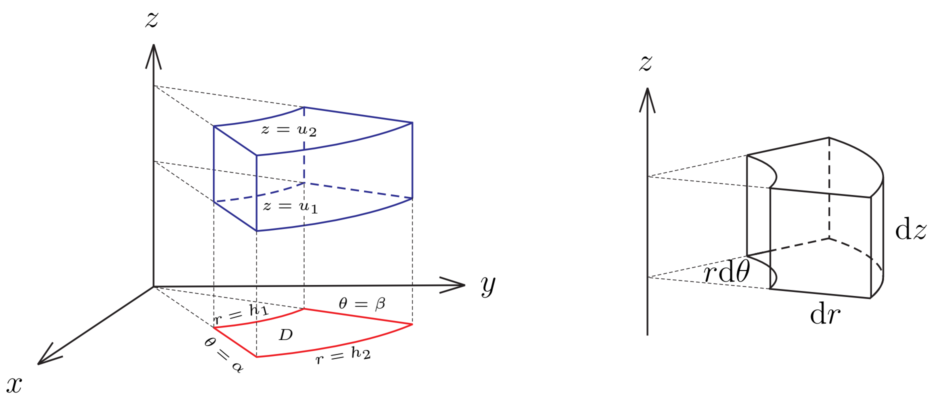

Figure 2: Cylindrical coordinate system.

Example 3. Evaluate $A = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x.$

Solution.

15N74CTcQPo

柱坐標的一個經典應用是可以處理三個圓柱垂直相交之體積。除了要會列式，實際計算也有技巧需要體會。

Exercise (page 1044). Find the volume of the solid enclosed by the three cylinders $x^2 + y^2 = 1$, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$.

Solution. The volume is ______________.

We leave the calculation of this integral as an Exercise. (有修微積分探究課的同學，將計算的結果補在第 22 頁的空白處。)

15.8 Triple Integrals in Spherical Coordinates, page 1045

Goal: Define and compute triple integrals in spherical coordinates.

y3Hblr0s_z0

Spherical Coordinates, page 1045

The *spherical coordinates* (ρ, θ, ϕ) (球坐標) of a point P in space are shown in Figure 1, where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP. Note that we assume $\rho \geq 0$ and $0 \leq \phi \leq \pi$.

球坐標也是另一個常見的坐標系，首先要認識 (ρ, θ, ϕ) 這三個參數對應的意義。注意到這裡設定的 ρ 非負，而 ϕ 的範圍選取是介在 0 到 π 之間。至於 θ 與極坐標的 θ 概念一致，比較可以根據問題而靈活地選取。

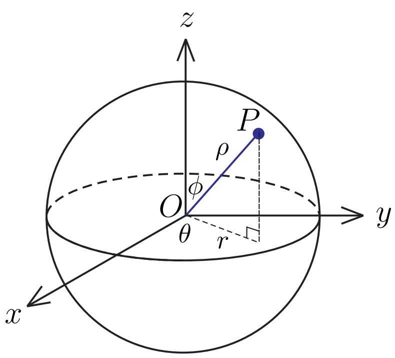

Figure 1: Spherical coordinate system.

☐ 不同的書籍或文章會用不同的記號 (有的用 r 而非 ρ) 與定義方式 (ϕ 的取法不同)。

The spherical coordinate system is useful in problems where there is symmetry about a point, and the origin is placed at this point. Figure 2 shows the surfaces of $\rho = c, \theta = c$, and $\phi = c$.

我們可以從坐標函數爲常數來理解球坐標，ρ 爲常數表示所有到坐標中心的距離爲定值之集合，即爲球；θ 爲常數時，因爲 ρ 非負，而 $0 \leq \phi \leq \pi$ 之下，會得到半平面；ϕ 爲常數時，因爲 ρ 非負，而 θ 任意，所以得到半錐。

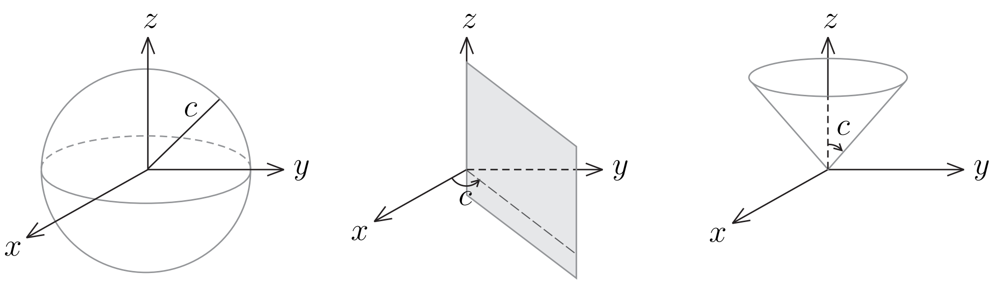

Figure 2: (a) $\rho = c$ is a sphere. (b) $\theta = c$ is a half-plane. (c) $\phi = c$ is a half-cone.

Relations between rectangular coordinates and spherical coordinates are

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho\cos\phi.$$

$$\rho^2 = x^2 + y^2 + z^2, \qquad \tan\theta = \frac{y}{x}, \qquad \tan^2\phi = \frac{x^2 + y^2}{z^2}.$$

直角坐標與球坐標之間的轉換關係也必須確實理解，重積分有時候會利用這些關係進行轉換。

PoZn6XjEybE

例題示範如何將指定的點對於球坐標與直角坐標之間的轉換。

Example 1 (page 1046)**.** The point $(2, \frac{\pi}{4}, \frac{\pi}{3})$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

Solution.

Example 2 (page 1046)**.** The point $(0, 2\sqrt{3}, -2)$ is given in rectangular coordinates. Find spherical coordinates for this point.

Solution.

Evaluating Triple Integrals with Spherical Coordinates, page 1047

zeNGpnvuQJw

這裡要建立球坐標系統下三重積分如何轉換成三次積分，重點是要確立球體積元素的表達。想清楚爲什麼要多乘上 $\rho^2 \sin\phi$ 這個倍率。

如前面的註解所述， 有些文獻使用球坐標系統的 θ, ϕ 代表不同的角度，所以不同的系統對應到的體積元素表達也會不一樣，要注意系統的一致性。

Consider the *spherical wedge* $E = \{(\rho, \theta, \phi) | a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$, where $a \ge 0$ and $\beta - \alpha \le 2\pi$, and $d - c \le \pi$.

(1) Divide E equally into E_{ijk} by $\rho = \rho_i, \theta = \theta_j$, and $\phi = \phi_k$.

(2) E_{ijk} is approximately a rectangular box with dimensions $\Delta\rho, \rho_i\Delta\phi$, and $\rho_i \sin\phi_k \Delta\theta$. So an approximation to the volume of E_{ijk} is given by

$$\Delta V_{ijk} \approx (\Delta\rho)(\rho_i\Delta\phi)(\rho_i \sin\phi_k\Delta\theta) = \rho_i^2 \sin\phi_k \Delta\rho\Delta\theta\Delta\phi.$$

In fact, by the Mean Value Theorem (see the Appendix), the volume of E_{ijk} is given exactly by

$$\Delta V_{ijk} = \tilde{\rho}_i^2 \sin\tilde{\phi}_k \Delta\rho\Delta\theta\Delta\phi,$$

where $(\tilde{\rho}_i, \tilde{\theta}_j, \tilde{\phi}_k)$ is some point in E_{ijk}. Let $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ be the rectangular coordinates of the sample point $(\tilde{\rho}_i, \tilde{\theta}_j, \tilde{\phi}_k)$.

(3) We get the Riemann sum

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\tilde{\rho}_i \sin\tilde{\phi}_k \cos\tilde{\theta}_j, \tilde{\rho}_i \sin\tilde{\phi}_k \sin\tilde{\theta}_j, \tilde{\rho}_i \cos\phi_j)\tilde{\rho}_i^2 \sin\tilde{\phi}_k \Delta\rho\Delta\theta\Delta\phi.$$

(4) When $l, m, n \to \infty$, we get the *formula for triple integration in spherical coordinates*:

$$\iiint_E f \, \mathrm{d}V = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\phi.$$

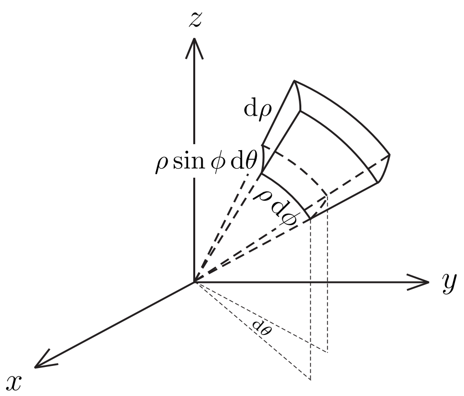

Figure 3: Volume element of the spherical coordinate system.

This formula can be extended to include more general spherical regions such as $E = \{(\rho, \theta, \phi) | \alpha \le \theta \le \beta, c \le \phi \le d, g_1(\theta, \phi) \le \rho \le g_2(\theta, \phi)\}$, and in this case the triple integration will be

$$\iiint_E f \, \mathrm{d}V = \int_c^d \int_\alpha^\beta \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\phi.$$

Example 3 (page 1048)**.** Evaluate $\iiint_B \mathrm{e}^{(x^2+y^2+z^2)^{\frac{3}{2}}} \, \mathrm{d}V$, where B is the unit ball: $B = \{(x, y, z) | x^2 + y^2 + z^2 \le 1\}$.

EVIObX_1TCE

例題示範用球坐標系處理三重積分。因爲積分的上下限都是常數，而且函數是分離變數的型式，所以可以各自處理積分再相乘。

Solution.

Example 4 (page 1048)**.** Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

k_8G_92iRM0

當球心與坐標中心不一致的時候，三重積分轉換成三次積分下的實體邊界就帶有變數。使用球坐標系統時，必須把球坐標的關係式寫下，才能呼應其體積元素。

Solution.

Appendix (Proof of the following statement)

N1OaKYBkSgs

在建立球坐標系統下的重積分之體積元素時，有一步驟是要利用均值定理以確定存在一個樣本點讓體積元素可以表示成 $\tilde{\rho}^2 \sin\tilde{\phi}\Delta\rho\Delta\theta\Delta\phi$，這裡就是要補充說明爲什麼的確有這一個樣本點。

"For all spherical wedge E, there exists $(\tilde{\rho}, \tilde{\theta}, \tilde{\phi})$ in E so that $\Delta V = \tilde{\rho}^2 \sin\tilde{\phi}\Delta\rho\Delta\theta\Delta\phi$."

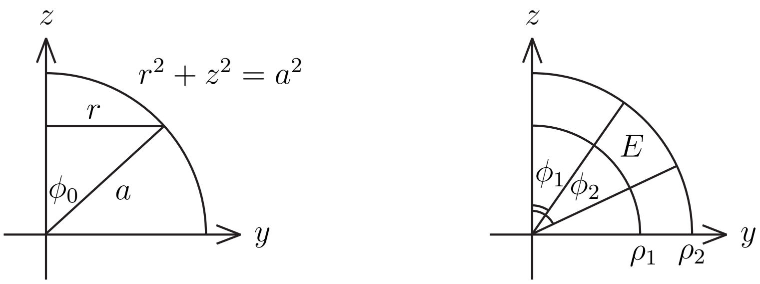

Figure 4: Volume element of the spherical coordinate system.

(a) Show that "the volume of the solid bounded above by the sphere $r^2+z^2=a^2$ and below by the cone $z = r\cot\phi_0, 0 < \phi_0 < \frac{\pi}{2}$, and $\theta_1 \le \theta \le \theta_2$, is $V = \frac{a^3\Delta\theta}{3}(1-\cos\phi_0)$, where $\Delta\theta = \theta_2 - \theta_1$". Using the cylindrical coordinates,

$$\begin{aligned} V &= \int_{\theta_1}^{\theta_2}\int_0^{a\sin\phi_0}\int_{r\cot\phi_0}^{\sqrt{a^2-r^2}} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta = \Delta\theta\int_0^{a\sin\phi_0}\left(r\sqrt{a^2-r^2}-r^2\cot\phi_0\right)\mathrm{d}r \\ &= \frac{\Delta\theta}{3}\left[-(a^2-r^2)^{\frac{3}{2}} - r^3\cot\phi_0\right]\Big|_{r=0}^{r=a\sin\phi_0} \\ &= \frac{\Delta\theta}{3}\left(-(a^2-a^2\sin^2\phi_0)^{\frac{3}{2}} + a^3 - a^3\sin^3\phi_0\cot\phi_0\right) \\ &= \frac{a^3\Delta\theta}{3}\left(1-\cos^3\phi_0 - \sin^2\phi_0\cos\phi_0\right) = \frac{a^3\Delta\theta}{3}(1-\cos\phi_0). \end{aligned}$$

(b) Show that the volume of the spherical wedge given by $\rho_1 \le \rho \le \rho_2, \theta_1 \le \theta \le \theta_2, \phi_1 \le \phi \le \phi_2$ is $\Delta V = \frac{\Delta\theta}{3}(\rho_2^3 - \rho_1^3)(\cos\phi_1 - \cos\phi_2)$. Denote V_{ij} by the volume of the region bounded by the sphere of radius ρ_i and the cone with angle ϕ_j, and θ from θ_1 to θ_2. Then we have

$$\begin{aligned} V &= (V_{22}-V_{21}) - (V_{12}-V_{11}) \\ &= \frac{\Delta\theta}{3}(\rho_2^3(1-\cos\phi_2) - \rho_2^3(1-\cos\phi_1) - \rho_1^3(1-\cos\phi_2) + \rho_1^3(1-\cos\phi_1)) \\ &= \frac{\Delta\theta}{3}(\rho_2^3-\rho_1^3)(\cos\phi_1 - \cos\phi_2). \end{aligned}$$

(c) By the Mean Value Theorem with $f(\rho) = \rho^3$, there exists some $\tilde{\rho} \in (\rho_1, \rho_2)$ such that $f(\rho_2) - f(\rho_1) = f'(\tilde{\rho})(\rho_2 - \rho_1) \Rightarrow \rho_2^3 - \rho_1^3 = 3\tilde{\rho}^2\Delta\rho$. Similarly, for $g(\phi) = \cos\phi$, there exists $\tilde{\phi} \in (\phi_1, \phi_2)$ such that $g(\phi_2) - g(\phi_1) = g'(\tilde{\phi})(\phi_2 - \phi_1) \Rightarrow \cos\phi_1 - \cos\phi_2 = \sin\tilde{\phi}\Delta\phi$. Hence for each spherical wedge E, there exists $(\tilde{\rho}, \tilde{\theta}, \tilde{\phi})$ in E such that

$$\Delta V_{ijk} = \tilde{\rho}_i^2 \sin\tilde{\phi}_k\Delta\rho\Delta\theta\Delta\phi.$$

15.9 Change Variables in Multiple Integrals, page 1052

Goal: Find relations of change of variable in double and triple integrals.
Recall that

YqNMwrSyIAg

這一節的目標是要建立一般坐標系統下的重積分。如此可以將之前所學所有坐標系之重積分觀念統整。在直角坐標系下，面積或體積元素的放大倍率是 1，在其它坐標系統的面積或體積元素的放大倍率是多少，就是這一節的重點。

(1) For a function of one variable $f(x)$, we have the Substitution Rule:

$$\int_a^b f(x)\,\mathrm{d}x = \int_c^d f(x(u))x'(u)\,\mathrm{d}u,$$

where $x = x(u)$ and $a = x(c), b = x(d)$.

(2) In section 15.4, we get the formula of double integrals in polar coordinates. Suppose that $x = r\cos\theta$, $y = r\sin\theta$, then

$$\iint_R f(x,y)\,\mathrm{d}A = \iint_S f(r\cos\theta, r\sin\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta,$$

where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane.

More generally, we consider a change of variables that is given by a C^1 and one-to-one *transformation* T form the uv-plane to the xy-plane (一次偏導數連續且一對一的坐標變換):

$$T(u,v) = (x,y),$$

9fBcxqgko1k

考慮坐標系之間的轉換，必須確定它是一對一的關係，而 C^1 變換是爲了可以使用微積分處理積分問題，所以要求比較好的變換關係。

where x and y are related to u and v by the equations

$$T : \begin{cases} x = x(u,v) \\ y = y(u,v), \end{cases} \qquad T^{-1} : \begin{cases} u = u(x,y) \\ v = v(x,y). \end{cases}$$

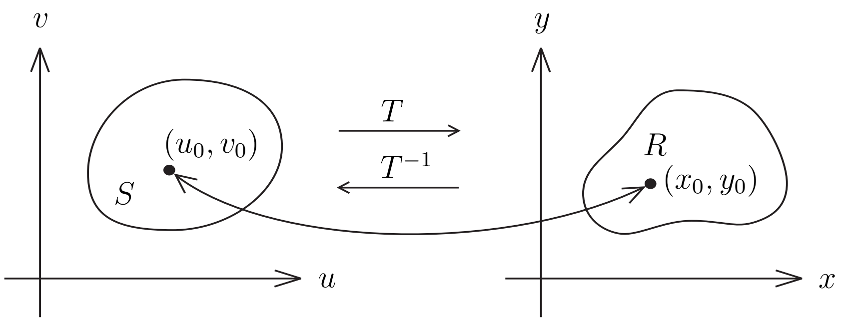

Figure 1: Transformation T and inverse transformation T^{-1}.

Definition 1 (page 1053)**.**

(a) The terminology C^1 means that $x(u,v)$ and $y(u,v)$ have continuous first-order partial derivatives.

(b) If $T(u_1,v_1) = (x_1,y_1)$, then (x_1,y_1) is called the *image* of (u_1,v_1).

(c) T is called *one-to-one* if no two points have the same image.

(d) T transforms S into a region R in the xy-plane called the *image of* S, consisting of the images of all points in S.

yMOE4tMR054

Example 2. Observe $\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x =$ Area of the region under $f(x)$.

觀察單變數函數積分的建構與等級的概念結合，只要掌握所有和 $\frac{1}{n}$ 有關的資訊，加總後求極限就會是區域面積，其它高階的「誤差」在加總取極限後是零。

Solution. Suppose that $f(x) \in C^1([a,b])$, which implies $|f'(x)| \le M$. Let $\Delta x = \frac{b-a}{n}$, then

$$\left| \int_a^b f(x)\,\mathrm{d}x - \sum_{i=1}^{n} f(x_i^*)\Delta x \right| \le \sum_{i=1}^{n} \left| \max_{[x_{i-1},x_i]} f(x) - \min_{[x_{i-1},x_i]} f(x) \right| \Delta x$$

$$\le \sum_{i=1}^{n} |f'(\xi_i)|(\Delta x)^2 = M \sum_{i=1}^{n} \frac{(b-a)^2}{n^2} = M \cdot \frac{(b-a)^2}{n} \to 0 \text{ as } n \to \infty.$$

So for integration, before we take summation, the $\frac{1}{n}$ part is the whole material. We can ignore higher order term such as $\frac{1}{n^2}$ because it tends to zero after summation and n tends to infinity.

VPKaCQ7fIyM

這一部份是要仔細說明在 C^1 且一對一的坐標變換下，如何確實地抓出對於每一個變量的「線性部份」，若要考慮面積，則對於每一個變量的線性增長所得到的平行四邊形面積，就是它的主要部份，其它誤差都是高階無窮小量。而線性增長的關係，我們就用雅可比 (Jacobi) 這個符號記錄。

Now we will see how a change of variables affects a double integral. We start with a small rectangle S in the uv-plane whose lower corner is the point (u_0, v_0) and whose dimensions are Δu and Δv. The image of S is a region R in the xy-plane, one of whose boundary points

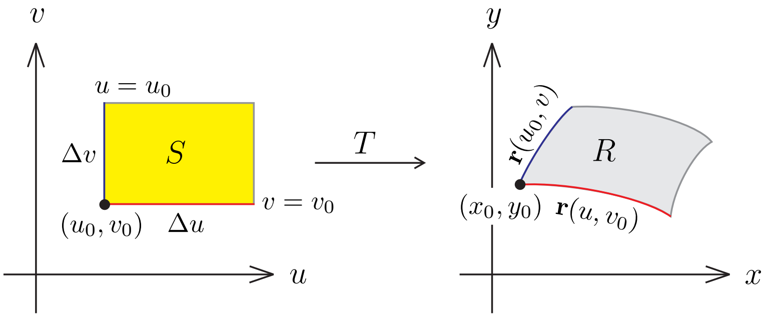

Figure 2: Transformation T from a rectangle S to a region R.

is $(x_0, y_0) = T(u_0, v_0)$. The vector $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j}$ is the position vector of the image of the point (u,v). The equation of the lower side of S is $v = v_0$, whose image curve is given by the vector function $\mathbf{r}(u, v_0)$. The tangent vector at (x_0, y_0) to this image curve is

$$\mathbf{r}_u = x_u(u_0, v_0)\,\mathbf{i} + y_u(u_0, v_0)\,\mathbf{j}$$

Similarly, the tangent vector at (x_0, y_0) to the image curve of the left side of S (namely, $u = u_0$) is

$$\mathbf{r}_v = x_v(u_0, v_0)\,\mathbf{i} + y_v(u_0, v_0)\,\mathbf{j}$$

We can approximate the image region $R = T(S)$ by a parallelogram determined by the secant vectors

$$\mathbf{a} = \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \qquad \text{and} \qquad \mathbf{b} = \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0).$$

Since

$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u}$$

and so

$$\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u \mathbf{r}_u \quad \text{and} \quad \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v \mathbf{r}_v.$$

This means that we can approximate R by a parallelogram determined by the vectors $\Delta u \mathbf{r}_u$ and $\Delta v \mathbf{r}_v$. Therefore we can approximate the area of R by the area of this parallelogram

$$|(\Delta u)\mathbf{r}_u \times (\Delta v)\mathbf{r}_v| = |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \Delta u \Delta v.$$

Definition 3 (page 1055)**.** The *Jacobian* of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

將坐標變換下線性增長的面積關係，用雅可比 (Jacobi) 這個符號記錄，就是之前所學一次偏微分所形成的矩陣之行列式。這裡我們討論的面積元素 dA 是沒有方向性的，所以直接把雅可比行列式加上絕對值。

With this notation we can get $\Delta A \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u \Delta v$, where the Jacobian is evaluated at (u_0, v_0).

For the general region S in the uv-plane we divide S into rectangles S_{ij} and call their images in the xy-plane R_{ij}. Applying the approximation to each R_{ij}, we approximate the double integral of f over R as follows:

$$\begin{aligned} \iint_R f(x, y)\,\mathrm{d}A &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_i)\Delta A \\ &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \left(f(x(u_i, v_i), y(u_i, v_i)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|_{(u_i,v_i)} \Delta u \Delta v + \text{H.O.T.} \right) \\ &= \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v. \end{aligned}$$

Change to Variables in a Double Integral (page 1056)**.** *Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then*

$$\iint_R f(x, y)\,\mathrm{d}A = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v.$$

這裡總結坐標變換下的重積分面積轉換式，兩組坐標系之下面積的增長倍率，是由坐標變換的一次偏微分式的行列式取絕對值而決定。

vcBn8JNspDQ

這個函數若要直接對於 x 或 y 積分基本上窒礙難行, 所以不妨把 $x+y$ 設定成一個新的變數, 把 $x-y$ 設定成另一個變數，至少函數變得清爽之下就有機會處理積分。另一方面，必須把原先的積分區域都要逐段翻譯成新坐標系之下的關係。在重積分的時候，除了函數替換與區域轉換外，記得雅可比行列式的絕對值要補上去。

因爲坐標變換之間的一次偏微分矩陣互爲反矩陣，所以行列式值互爲倒數，應善用這個關係式。

Example 4 (page 1058)**.** Evaluate the integral $\iint_R e^{\frac{x+y}{x-y}}\,dA$, where R is the trapezoidal region with vertices $(1,0),(2,0),(0,-2),$ and $(0,-1)$.

Solution.

XdSJmjKNZ6o

積分的區域是一個斜橢圓，或許有人想到可以利用坐標旋轉的方式將區域對於新坐標而言是橢圓的標準式，再進行重積分，這個方法是可行的，只是需要比較多的計算。若只是找重積分的值，在例題的示範中，直接對某個變數配方法，再重新進行變數變換，這時直接把橢圓修正成圓形 (線性變換)， 然後確定變換後的面積倍率，就可以處理重積分了。

Example 5. Evaluate $\iint_{x^2+xy+y^2\le 1} e^{-(x^2+xy+y^2)}\,dA$.

Solution.

Triple Integrals, page 1059

The *Jacobian* of the transformation T is the following 3×3 determinant:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

We have the following formula for triple integrals:

$$\iint_R f(x,y,z)\,\mathrm{d}V = \iint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w.$$

Example 6. Elliptic cylindrical coordinate system is

$$x = ar\cos\theta, \qquad y = br\sin\theta, \qquad \tilde{z} = cz,$$

where $a, b, c > 0$ are constants. The volume element is ____________________.

Example 7. Ellipsoidal coordinate system is

$$x = a\rho\sin\phi\cos\theta, \qquad y = b\rho\sin\phi\sin\theta, \qquad z = c\rho\cos\phi,$$

where $a, b, c > 0$ are constants. The volume element is ____________________.

JiEJcFt82X0

我們可以對於之前討論的概念對維度進行推廣，這裡就直接列出三重積分的坐標變換轉換公式，其線性的增長倍率也是由坐標變換的一次偏微分所形成的矩陣所決定。於是之前所學的柱坐標與球坐標之體積元素，各位應再與這個觀念結合起來。更進一步地，我們可以把之前學到的坐標系，對於各變量做常數倍的變換，於是有橢柱坐標或橢球坐標等。

處理問題的時候，坐標系只是一個幫助我們解決問題的工具，而坐標系的選取可以有多樣性與任意性，遇到問題時，都應以多方嘗試每種系統下的優缺點。