Chapter 14 Partial Derivatives

14.1 Functions of Several Variables, page 888

Definition 1 (page 888)**.** A *function f of two variables* (雙變數函數) is a rule that assigns to each ordered pair of real numbers (x, y) in a set $D \subset \mathbb{R}^2$ a unique real number denoted by $f(x, y)$. The set D is the *domain* (定義域) of f and its *range* (值域) is the set of values that f takes on, that is, $\{f(x, y)|(x, y) \in D\}$.

GcrflrNwuq8

這一章開始正式進入多變數函數的微積分理論，一開始還是要先介紹二變數函數的定義域、對應域等相關意義。

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y). The variables x and y are *independent variables* (獨立變數) and z is the *dependent variable* (依賴變數).

Example 2 (page 888–889)**.**

(a) Function: $f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$. Domain: $D = \{(x, y)|x + y + 1 \geq 0, x \neq 1\}$.

(b) Function: $g(x, y) = x \ln(y^2 - x)$. Domain: $D = \{(x, y)|x < y^2\}$.

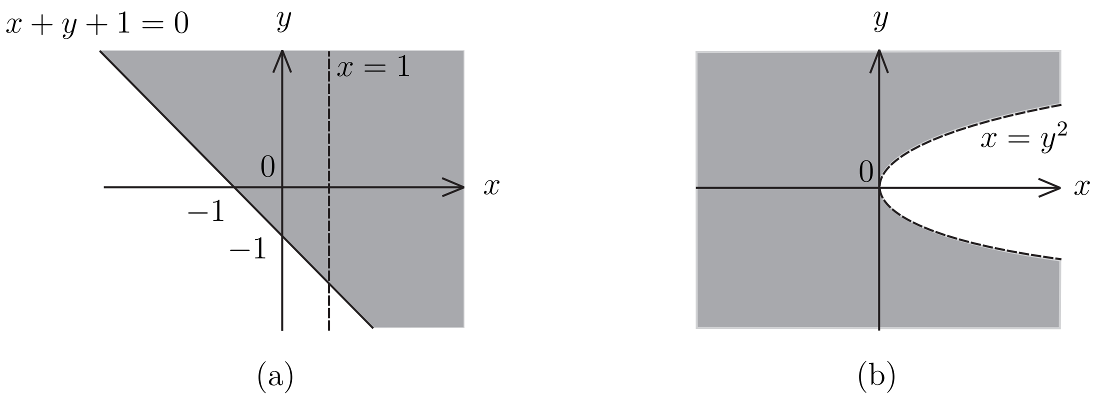

Figure 1: (a) Domain of $f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$. (b) Domain of $g(x, y) = x \ln(y^2 - x)$.

Exercise. Find and sketch the domain of the function $f(x, y) = \sin^{-1}(x^2 + y^2 - 2)$.

Graphs, page 890

One way of visualizing the behavior of a function of two variables is to consider its graph.

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認識函數的一種方法是畫圖，對於雙變數函數而言，它的圖形就必須放到三度空間當中理解。

Definition 3 (page 890)**.** If f is a function of two variables with domain D, then the *graph* of f (圖形) is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

The graph of a function f of two variables is a surface S with equation $z = f(x, y)$. We can visualize the graph S of f as lying directly above or below its domain D in the xy-plane.

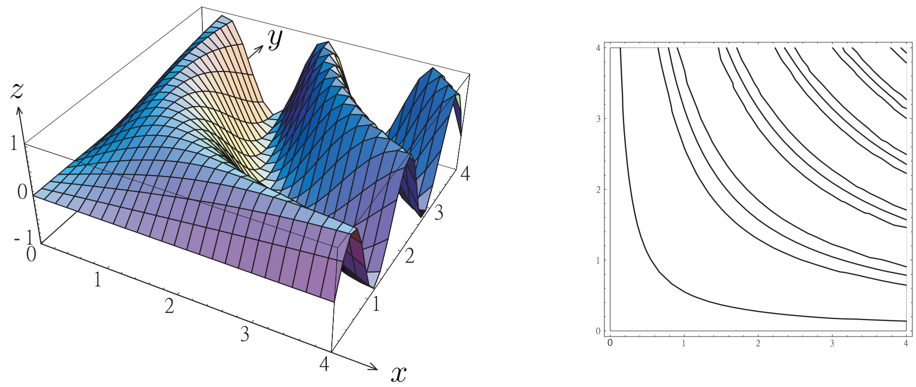

Figure 2: The graph of $f(x, y) = \sin xy, 0 \leq x \leq 4, 0 \leq y \leq 4$ and its level curves.

Level Curves, page 893

另外一種認識二變數函數的方法是在定義域上畫出它的等高線，就像平常在地圖上看到的那些線條，也可以解讀出地勢（函數）的地伏與變化。

Another method for visualizing functions, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form *contour lines* (等高線、輪廓線), or *level curves* (等位線).

Definition 4 (page 893)**.** The *level curves* of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy-plane. The surface is steep where the level curves are close together. It is somewhat flatter where they are farther apart.

☐ 等高線、等壓線、等溫線。

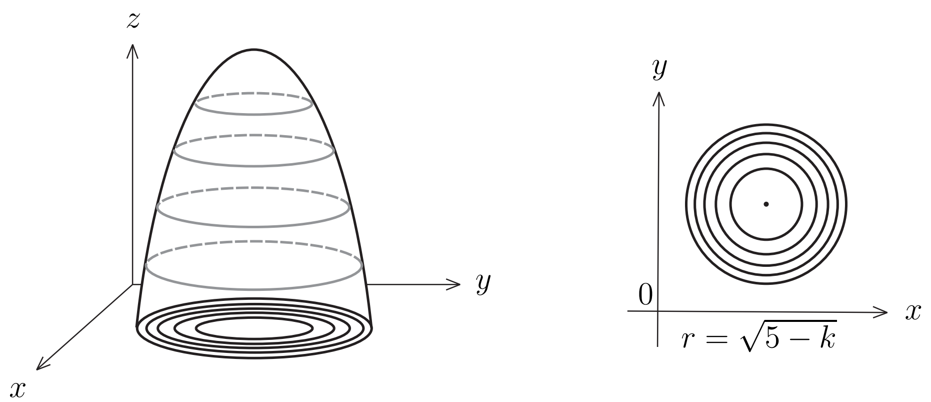

Figure 3: Level curves of the function $f(x, y) = 5 - (x - 3)^2 - (y - 3)^2$.

Exercise (page 902)**.** Match the function (a),(b),(c) with its graph (A),(B),(C) and its contour map (I), (II), (III). Give reasons for your choices.

(a) $f(x, y) = \sin x - \sin y$ (b) $g(x, y) = \dfrac{x - y}{1 + x^2 + y^2}$ (c) $h(x, y) = e^x \cos y$.

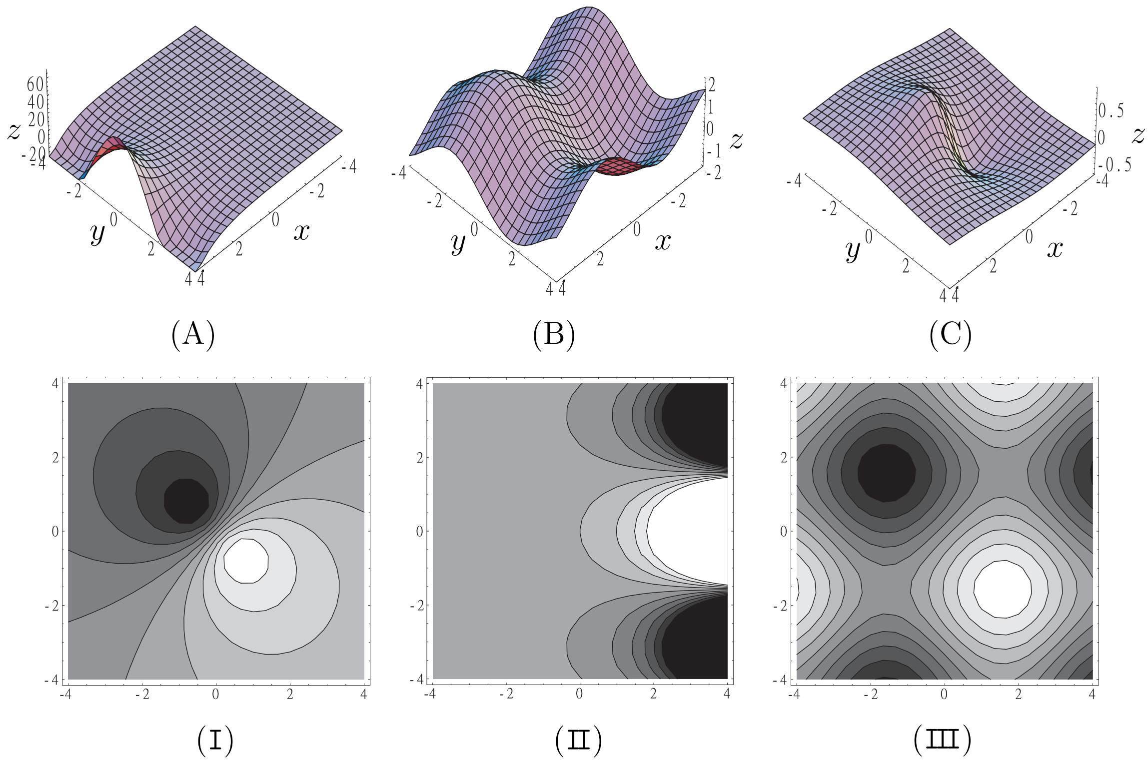

Figure 4: Match functions, graphs, and contour maps.

Functions of Three or More Variables, page 89

A *function of three variables* (三變數函數), f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z) \in \mathbb{R}$. For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time, so we could write $T = f(x, y, t)$.

微積分課程中主要以二變數函數與三變數函數討論，而一些理論實際上與維度無關，故可推廣至 n-變數函數。

In general, a *function of* n *variables* (n-變數函數) is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. Sometimes are will use vector notation to write such functions more compactly: If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, we often write $f(\mathbf{x})$ in place of $f(x_1, x_2, \ldots, x_n)$.

14.2 Limits and Continuity, page 903

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二變數函數在一點的極限，給定誤差 $\varepsilon > 0$ 後，必須要找到一個範圍內所有點的函數值與極限值的差要在誤差內。

Definition 1 (page 904)**.** Let f be a function of two variables whose domain D includes points arbitrary close to (a, b). Then we say that the *limit of* $f(x, y)$ *as* (x, y) *approaches* (a, b) is L (函數 $f(x, y)$ 靠近 (a, b) 的極限值是 L) and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x, y) - L| < \varepsilon$.

Other notations for the limit are

$$\lim_{\substack{x\to a\\ y\to b}} f(x, y) = L \qquad \text{and} \qquad f(x, y) \to L \text{ as } (x, y) \to (a, b).$$

因爲二變數函數的定義域是一個區域，會有各種可能靠近 (a, b) 的方法，極限存在的意思是不論用何種方式靠近，函數值都要接近同一個數字。

The definition refers only to the *distance* between (x, y) and (a, b). It does not refer to the direction of approach. Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b). Therefore, we get

Property 2 (page 905)**.** *If* $f(x, y) \to L_1$ *as* $(x, y) \to (a, b)$ *along a path* C_1 *and* $f(x, y) \to L_2$ *as* $(x, y) \to (a, b)$ *along a path* C_2*, where* $L_1 \neq L_2$*, then* $\lim_{(x,y)\to(a,b)} f(x, y)$ *does not exist.*

☐ 多變數函數極限存在 $\Leftrightarrow$ 以「任何路徑」靠近都要接近一個明確的值。

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Example 3 (page 905)**.** Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

Solution.

以下幾個例子是要仔細研究多變數函數的極限與單變數函數的極限之間的差別。對於第一個例子而言， 只要沿著 x-軸與 y-軸靠近 $(0, 0)$ 就會得到不同的極限值。第二個例子的結論是：雖然沿著 x-軸與 y-軸靠近 $(0, 0)$ 極限值一樣，但是從 45 度角切入原點的極限值不同，所以極限仍然不存在。

Example 4 (page 906)**.** If $f(x, y) = \frac{xy}{x^2+y^2}$, does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist?

Solution.

Example 5 (page 906)**.** If $f(x,y) = \frac{xy^2}{x^2+y^4}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Solution.

HLBkQmXy26Q

從各種方向直接切入原點的極限值縱使都一樣，但仍然不保證二變數函數的極限值存在，因爲選取路徑的方法可以很任意，像這個例子，考慮拋物路徑的方式又會有不同的極限值，所以二變數函數的極限不存在。

☐ 上例得知如果只有以「各種角度」直線靠近一個明確的值, 極限仍有可能不存在。

We can use polar coordinates to find the limit. Note that if (r,θ) are polar coordinates of the point (x,y) with $r \geq 0$, then $r \to 0^+$ as $(x,y) \to (0,0)$.

用極坐標的方式改寫函數再搭配夾擠定理求二變數函數的極限是一個可行的方法。

Example 6 (page 896)**.** Find $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ if it exists.

Solution.

Continuity, page 907

Definition 7 (page 908)**.** A function f of two variables is called *continuous at* (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b) = f\left(\lim_{(x,y)\to(a,b)} x, \lim_{(x,y)\to(a,b)} y\right).$$

We say f is *continuous on* D if f is continuous at every point (a,b) in D.

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函數的連續性與單變數函數的概念相近，可想成是函數與極限可以交換的意思。

A *polynomial function of two variables* (or *polynomial* 二變數多項式, for short) is a sum of terms of the form cx^my^n, where c is a constant and m and n are nonnegative integers. A *rational function* (有理函數) is a ratio of polynomials. All polynomials are continuous on $\mathbb{R}^2$. Any rational function is continuous on its domain because it is a quotient of continuous functions.

Example 8 (page 908)**.** Where is the function $f(x,y)=\frac{x^2-y^2}{x^2+y^2}$ continuous?

Solution. The function $f(x,y)$ is discontinuous at $(0,0)$ because it is not defined there. Since $f(x,y)$ is a rational function, it is continuous on its domain $D=\{(x,y)|(x,y)\neq(0,0)\}$.

Example 9 (page 908)**.** Let

$$f(x,y)=\begin{cases}\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}.$$

Here $f(x,y)$ is defined at $(0,0)$ but $f(x,y)$ is still discontinuous there because $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist. (See **Example 3**.)

Example 10. Let

$$f(x,y)=\begin{cases}\frac{3x^2y}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}.$$

We know $f(x,y)$ is continuous for $(x,y)\neq(0,0)$ since it is equal to a rational function there. From **Example 6**, we have

Therefore, ______________________________, and so it is continuous on ___.

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Property 11 (page 909)**.** *If f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f, then the composite function $h=g\circ f$ defined by $h(x,y)=g(f(x,y))$ is also a continuous function.*

這裡列出合成函數的連續性。例題的函數在多變數函數理論中算是蠻重要的函數。

Example 12 (page 909)**.** Where is the function $h(x,y)=\tan^{-1}\left(\frac{y}{x}\right)$ continuous?

Solution. The function $f(x,y)=\frac{y}{x}$ is a rational function and therefore continuous except on ________. The function $g(t)=\tan^{-1}t$ is continuous everywhere, so the composition function $h(x,y)=g(f(x,y))=\tan^{-1}\left(\frac{y}{x}\right)$ is continuous except where ________.

Functions of Three or More Variables, page 909

超過三個以上的變數之函數理論也可以同理類推。

Everything that we have done in this section can be extended to functions of three or more variables. The notation

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z)=L$$

means that the values of $f(x,y,z)$ approach the number L as the point (x,y,z) approaches the point (a,b,c) along *any* path in the domain of f. The function f is *continuous* at (a,b,c) if

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z)=f(a,b,c)=f\left(\lim_{(x,y,z)\to(a,b,c)} x,\ \lim_{(x,y,z)\to(a,b,c)} y,\ \lim_{(x,y,z)\to(a,b,c)} z\right).$$

For a function of n variables, we can write these definitions in a single compact form by vector notation. For instance, let $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, $\mathbf{a}=(a_1,a_2,\ldots,a_n)$, and $f(\mathbf{x})$ is a function of n variable, The function f is *continuous* at $\mathbf{a}$ if

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})=f(\mathbf{a})=f\left(\lim_{\mathbf{x}\to\mathbf{a}}\mathbf{x}\right).$$

14.3 Partial Derivative, page 911

Definition 1 (page 913)**.** If f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y = y_0$, then $g(x) = f(x, y_0)$ is a function of a single variable x. If $g(x)$ has a derivative at $x = x_0$, then we call it the *partial derivative* (偏導數) *of f with respect to x at* (x_0, y_0) and denote it by $f_x(x_0, y_0)$. Thus

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若要研究二變數函數, 當中的一個思想是: 是否能夠利用單變數函數所學到的東西來理解多變數。雖然在極限的討論似乎處處受挫, 但是在微分的層級下, 在「某些」情況又可以如此處理。接下來的兩個單元就是要研究這個問題。

$$f_x(x_0, y_0) = g'(x_0) = \lim_{h\to 0} \frac{g(x_0 + h) - g(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}.$$

Similarly, the *partial derivative* (偏導數) *of f with respect to y at* (x_0, y_0) and denote it by $f_y(x_0, y_0)$, is obtained by keeping x fixed, say $x = x_0$, and finding the ordinary derivative at $y = y_0$ of the function $\tilde{g}(y) = f(x_0, y)$:

$$f_y(x_0, y_0) = \tilde{g}'(y_0) = \lim_{h\to 0} \frac{\tilde{g}(y_0 + h) - \tilde{g}(y_0)}{h} = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Definition 2 (page 913)**.** If f is a function of two variables, its *partial derivatives* (偏導函數) are the functions f_x and f_y defined by

偏導數的意思就是固定其它的變數, 讓函數變成單變數函數, 去觀察函數對於這個變數的變化。

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h},$$
$$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

Notations for Partial Derivatives. *If $z = f(x, y)$, we write*

偏導數的記號有很多種, 也都很常見, 最後兩個記號與方向導數有關, 可以到 14.6 的時候再一起統整。

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_x f = D_1 f,$$
$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_y f = D_2 f.$$

Rule for Finding Partial Derivative of $z = f(x, y)$.

(1) *To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.*

(2) *To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.*

☐ 對某變數求偏導, 固定其他變數, 使其爲單變數函數, 再計算導數。

Example 3 (page 914)**.** If $f(x, y) = x^3 + x^2y^3 - 2y^2$, then

(a) $f_x(x, y) =$

$f_x(2, 1) =$

(b) $f_y(x, y) =$

$f_y(2, 1) =$

Interpretations of Partial Derivatives, page 915

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回想單變數函數在一點求導，幾何意義就是在問函數圖形在那一點的切線斜率。而偏導數的幾何意義也可以類似地去想它，二變數函數圖形與平面 $x = x_0$ 或是 $y = y_0$ 相交，得到單變數函數的圖形，然後去問在該點的切線斜率。

The partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ can be interpreted geometrically as the slopes of the tangent lines at $P(x_0, y_0, f(x_0, y_0))$ to the trace C_1 and C_2 of the surface S in the planes $y = y_0$ and $x = x_0$.

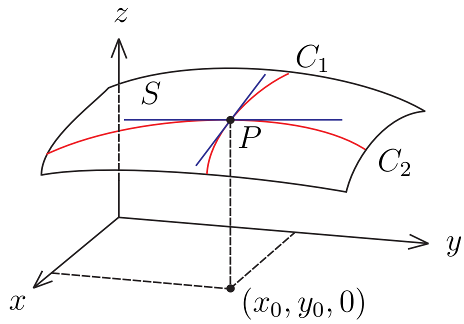

Figure 1: Geometric meaning of partial derivatives.

Example 4 (page 917)**.** If $f(x, y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution.

Example 5 (page 917)**.** Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation $x^3 + y^3 + z^3 + 6xyz = 1$.

Solution.

Exercise (page 927)**.** If $f(x, y) = \dfrac{x\,\mathrm{e}^{\sin(x^2y)}}{(x^2+y^2)^{\frac{3}{2}}}$, find $f_x(1, 0)$.

Functions of More Than Two Variables, page 917

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認識多變數函數的偏導函數之記號。

If $z = f(x_1, x_2, \ldots, x_n)$ is a function of n variables, its partial derivative with respect to the i-th variable x_i is

$$\frac{\partial z}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}.$$

We also write

$$\frac{\partial z}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f.$$

Higher Derivatives, page 918

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x, (f_x)_y, (f_y)_x$, and $(f_y)_y$, which are called the *second partial derivatives* (二次偏導數) of f. If $z = f(x, y)$, we use the following notation:

高次偏導函數的記號要小心, 以函數爲主體, 由內而外的順序操作, 所以用下標註記的時候順序是由左至右, 用 ∂ 的符號註記時是由右至左。當函數不是很好的時候, 有可能偏微分的順序不同而會有不同的值。

$$
\begin{aligned}
(f_x)_x &= f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2},\\
(f_x)_y &= f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x},\\
(f_y)_x &= f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y},\\
(f_y)_y &= f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}.
\end{aligned}
$$

☐ 寫成下標的順序 f_{xy} 和寫成 $\frac{\partial^2 f}{\partial y \partial x}$ 的順序及其代表之意義必須注意。

Exercise. Let $r(x, y) = \sqrt{x^2 + y^2}$. For $(x, y) \neq (0, 0)$, compute $r_x, r_y, r_{xx}, r_{xy}, r_{yx}$, and r_{yy}.

Clairaut's Theorem (page 919)**.** *Suppose f is defined on a disk D that contains the point (x_0, y_0). If the functions f_{xy} and f_{yx} are both continuous on D, then*

定理告知: 當函數二次偏微分後的函數都要是連續函數時, 偏微分順序才可以交換。這一節的最後會有一個例子說明: 存在函數其二次偏微分的順序交換不同。

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

☐ 二次偏導函數 f_{xy} 與 f_{yx} 必須都是「連續函數」, 偏微分順序交換才會相等。

Exercise. Let $f(x, y) = \dfrac{x^3 - xy^2}{x^2 + y^2}$.

(a) Determine the value $f(0, 0)$ such that $f(x, y)$ is continuous at $(0, 0)$.

(b) Find $f_x(x, y), f_x(x, y), f_x(0, 0)$ and $f_y(0, 0)$.

(c) Compute $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.

Partial Differential Equations, page 920

Partial derivatives occur in *partial differential equations* (偏微分方程) that express certain physical laws. For instance,

偏微分方程式的研究是近代數學討論的一個重點, 不論方程式在工程科學上有重要的應用, 數學上方程式解的存在、唯一、正則性也是廣泛地被探討。

(a) $u = u(x, y), \Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0$: Laplace's equation (拉普拉斯方程).

(b) $u = u(t, x), \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$: heat equation (熱傳導方程).

(c) $u = u(t, x), \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$: wave equation (波動方程).

YvOYwcKytq8

這個例子將檢視偏導數的定義，還有二次偏微分順序交換的異同。由於這個函數的二次偏微分在原點不連續，即 Clairaut 定理的條件不成立，而這個例子告知順序互換在函數不太好的時候的確有差。

Example 6 (page 927)**.** Let $f(x,y) = \begin{cases} \frac{x^3y-xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

(a) Find $f_x(x,y)$ and $f_y(x,y)$ when $(x,y) \neq (0,0)$.

(b) Find $f_x(0,0)$ and $f_y(0,0)$.

(c) Find $f_{xy}(x,y)$ and $f_{yx}(x,y)$ when $(x,y) \neq (0,0)$.

(d) Find $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

(e) Do the results of (c) and (d) contradict Clairaut's Theorem?

Solution.

(a) Direct computation gives

$$f_x(x,y) = \frac{(x^2+y^2)(3x^2y-y^3)-(x^3y-xy^3)(2x)}{(x^2+y^2)^2} = \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}$$

$$f_y(x,y) =$$

(b) By definition, we have

$$f_x(0,0) =$$

$$f_y(0,0) =$$

(c) Direct computation gives

$$f_{xy}(x,y) = \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}$$
$$f_{yx}(x,y) = \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}.$$

(d) By definition, we have

$$f_{xy}(0,0) =$$

$$f_{yx}(0,0) =$$

(e) Results of (c) and (d) don't contradict to Clairaut's Theorem because both $f_{xy}(x,y)$ and $f_{yx}(x,y)$ are not continuous at $(0,0)$. We cant take path $C_1(x) = (x,0), x \neq 0$ and $C_2(y) = (0,y), y \neq 0$ to get $f_{xy}(x,y)|_{C_1(x)} = f_{yx}(x,y)|_{C_1(x)} \equiv 1$ and $f_{xy}(x,y)|_{C_2(y)} = f_{yx}(x,y)|_{C_2(y)} \equiv -1$. That is, $\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ and $\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ do <u>not</u> exist.

□ 分段函數求偏導，用定義計算。由 (d) 知，二次偏導函數順序交換不見得相等。

14.4 Tangent Planes and Linear Approximations, page 927

Tangent Planes, page 928

Definition 1 (page 928)**.** Suppose that a surface S has equation $z = f(x, y)$, where f has *continuous partial derivatives*, and let $P(x_0, y_0, z_0)$ be a point on S. Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at P. Then the *tangent plane* (切平面) to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2.

vKAHbCqczHg

函數圖形在一點的切平面是由坐標曲線在那一點的切向量所張出的平面。

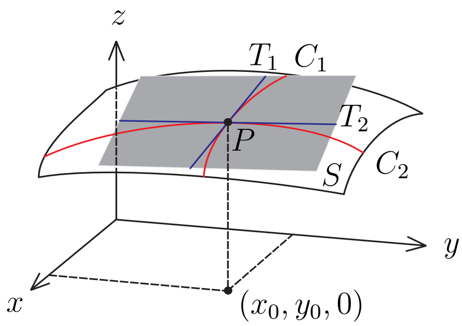

Figure 1: The tangent plane contains the tangent lines T_1 and T_2.

An equation of the tangent plane to the surface $z = f(x, y)$ at $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0), \quad \text{or} \qquad (點斜式)$$
$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0 \qquad (用法向量看待)$$

函數圖形在一點的切平面表示法可以由兩種觀點去思考，一種是由平面中的直線點斜式類推，另一種是由法向量的觀點出發，用兩個切向量作外積而得法向量。

Remark 2. Since tangent vectors to C_1 and C_2 at P are $\mathbf{e}_1 = 1\,\mathbf{i} + 0\,\mathbf{j} + f_x(x_0, y_0)\,\mathbf{k}$ and $\mathbf{e}_2 = 0\,\mathbf{i} + 1\,\mathbf{j} + f_y(x_0, y_0)\,\mathbf{k}$, a normal vector of the tangent plane is

$$\begin{aligned} \mathbf{n} = \mathbf{e}_1 \times \mathbf{e}_2 &= -f_x(x_0, y_0)\,\mathbf{i} - f_y(x_0, y_0)\,\mathbf{j} + 1\,\mathbf{k} \\ &/\!\!/ \; f_x(x_0, y_0)\,\mathbf{i} + f_y(x_0, y_0)\,\mathbf{j} - 1\,\mathbf{k}. \end{aligned}$$

☐ 若函數具有「連續的偏導數」(f_x 與 f_y 是連續函數)，才有切平面。

Example 3. Find the equation of the tangent plane of the surface $z = e^{x-y}$ at the point $P(1, 1, 1)$.

Solution.

Linear Approximations, page 929

WMSKqix7Wp8

函數在一點的線性估計就是用切平面上 z 分量的取值估計原函數的值。在單變數函數理論, 求導與微分同義, 但是在多變數函數理論, 偏導與可微分將分成兩個不同的概念。

Definition 4 (page 929)**.** An equation of the tangent plane to the graph of the function $z = f(x, y)$ at $P(x_0, y_0, z_0)$ is $z - z_0 = z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. The linear function whose graph is this tangent plane, namely,

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

is called *linearization* (線性化) of f at (x_0, y_0) and the approximation

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{1}$$

is called the *linear approximation* (線性估計) or *tangent plane approximation* of f at (x_0, y_0).

這個例子是說明: 雖然在 x 與 y 方向有好的線性估計, 但是兩者線性組合出來的值可能無法好好的近似函數。

Example 5 (page 930)**.** Consider the function $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

(a) $f_x(0, 0) =$

(b) $f_y(0, 0) =$

(c) We take the path $C_1(t) = (t, t), t \neq 0$, the function $f(x, y)|_{C_1(t)} =$

(d) A function of two variables can behave badly even through both of its partial derivatives exist. To rule out such behavior, we will define a *differentiable function* (可微分函數) of two variable.

F3ViScKLk38

二變數函數可微分的意思就是可以用切平面對應的線性函數做函數的線性估計。

Definition 6 (page 931)**.** If $z = f(x, y)$, then f is *differentiable* (可微分的) at (x_0, y_0) if $\Delta x = x - x_0, \Delta y = y - y_0$, then $f(x, y)$ satisfies

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{f(x, y) - f(x_0, y_0) - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0.$$

Sometimes it is hard to use the definition to check the differentiability of a function, but the next theorem provides a convenient sufficient condition for differentiability.

當二變數函數的偏微分是連續時才可以線性估計。

Theorem 7 (page 932)**.** *If the partial derivatives f_x and f_y exist near (x_0, y_0) and are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).*

☐ 函數具有連續偏導數, 切平面相應的線性函數才是好的線性估計。

☐ 多變數函數, 導數 (derivative) 與可微分 (differentiable) 兩者概念上有別。

Differentials, page 932

For a differentiable function of two variables, $z = f(x, y)$, we define the *differentials* (微分) $\mathrm{d}x$ and $\mathrm{d}y$ to be independent variables; that is, they can be given any values. Then the *differential* $\mathrm{d}z$, also called the *total differential* (全微分), is defined by

對於可微分函數來說，因爲它的線性估計有意義，所以特別把它的線性增長量用 $\mathrm{d}z$ 這個記號表示，稱爲全微分。

$$\mathrm{d}z = \mathrm{d}f = f_x(x,y)\,\mathrm{d}x + f_y(x,y)\,\mathrm{d}y = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y = \frac{\partial z}{\partial x}\,\mathrm{d}x + \frac{\partial z}{\partial y}\,\mathrm{d}y. \tag{2}$$

If we take $\mathrm{d}x = \Delta x = x - x_0$ and $\mathrm{d}y = \Delta y = y - y_0$ in (2), then the differential of z is $\mathrm{d}z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$, so in notation of differentials, the linear approximation (1) can be written as $f(x, y) \approx f(x_0, y_0) + \mathrm{d}z$.

Figure 2 shows the geometric interpretation of the differential $\mathrm{d}x$ and the increment Δz: $\mathrm{d}z$ represents the change in height of the tangent plane, whereas Δz represents the change in height of the surface $z = f(x, y)$ when (x, y) changes from (x_0, y_0) to $(x_0 + \Delta x, y_0 + \Delta y)$.

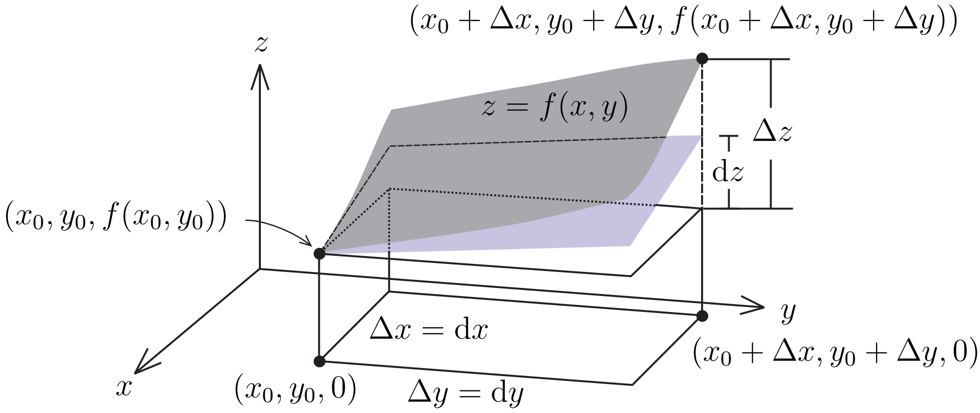

Figure 2: Geometric interpretation of the differential $\mathrm{d}z$ and the increment Δz.

Example 8 (page 933)**.** The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

Solution.

Functions of Three or More Variables, page 932

Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables.

SYgeaTX1B0k

這個例子也是重申可微分與線性估計之間的關係，這個函數在原點不是可微分的，所以函數在原點做出來切平面對應到的線性函數在其它方向來看無法確實呈現其線性估計。

Example 9. Let $f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(a) $f(x,y)$ is continuous at $(0,0)$ because

(b) $f_x(0,0) =$

(c) $f_y(0,0) =$

(d) For $(x,y) \neq (0,0)$, $\frac{\partial f}{\partial x} =$

(e) $\frac{\partial f}{\partial x}(x,y)$ is *not* continuous at $(0,0)$ because we take the path $C_1(t) = (t,t), t \neq 0$, then the function $f_x(x,y)|_{C_1(t)} =$

(f) Compute for $(x,y) \neq (0,0)$

$$f(x,y) - f(0,0) - f_x(0,0)x - f_y(0,0)y =$$

and take the path $C_1(x) = (x,x), x \neq 0$, we find

$$f(x,y) - f(0,0) - f_x(0,0)x - f_y(0,0)y|_{C_1(x)} =$$

(g) From (e) and (f), we know that $L(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y \equiv 0$ is *not* a good linear approximation of $f(x,y)$ at $(0,0)$.

14.5 The Chain Rule, page 937

The Chain Rule, Case 1 (page 938). *Suppose that $z = z(x, y)$ is a differentiable function of x and y, where $x = x(t)$ and $y = y(t)$ are both differentiable function of t. Then z is a differentiable function of t and*

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

pyUgdh2cf8A

鏈鎖律是二變數函數微分的重點，必須清楚所有變量之間的層級關係，確實討論其變化率。第一類型的鏈鎖律若要用幾何的方式去解釋，則是探討函數圖形上的曲線之高度的變化率。

☐ z 和 x, y 與 t 的關係式爲 $z(t) = z(x(t), y(t))$。

Example 1 (page 938). If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find $\frac{\mathrm{d}z}{\mathrm{d}t}$ when $t = 0$.

Solution.

Example 2. Find the second derivative $\frac{\mathrm{d}^2z}{\mathrm{d}t^2}$.

Solution.

The Chain Rule, Case 2 (page 939). *Suppose that $z = z(x, y)$ is a differentiable function of x and y, where $x = x(s, t)$ and $y = y(s, t)$ are differentiable functions of s and t. Then*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

B1grFNAmbcI

第二類的鏈鎖律討論層級是二變數與二變數之間的變化，在幾何意義上是在觀察函數在坐標變換下的轉換關係。

☐ z 與 x, y 和 x, t 的關係式爲 $z(s, t) = z(x(s, t), y(s, t))$。

Case 2 of the Chain Rule contains three types of variables: s and t are *independent variables*, x and y are *intermediate variables*, and z is the *dependent variable*.

Example 3. If $z = \mathrm{e}^x \sin y$, where $x = st^2$ and $y = s^2t$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution.

Example 4. Let $z = y + f(x^2 - y^2)$ and f be a differentiable function of single variable. Find $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$.

Solution.

The Chain Rule, General Version (page 940). *Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$, and each x_i is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and*

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial t_i}.$$

☐ 以上鏈鎖律爲合成函數的求導法則。

Coordinates Changes

hJppr1-yhQ8

直角坐標與極坐標之間的坐標變換關係非常重要，由此學習鏈鎖律之外，也可以看出很多現象。例如，兩種變數之間的一次偏微分關係，寫在矩陣型式時，互爲反矩陣。

In $\mathbb{R}^2$, denote (x, y) as the Cartesian coordinates and (r, θ) as the polar coordinates. We know relations between these coordinates are

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta. \end{cases} \qquad \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases}.$$

So we know $x = x(r, \theta), y = y(r, \theta)$ and $r = r(x, y), \theta = \theta(x, y)$, and hence

$$x = x(r(x, y), \theta(x, y)) \quad \text{and} \quad y = y(r(x, y), \theta(x, y)).$$

Since x and y are independent variables, we have

$$\left\{ \begin{array}{l} \\ \\ \\ \\ \end{array} \right. \qquad \Rightarrow$$

So partial derivatives of coordinates changes form inverse matrices. Now we check this relation by computing partial derivatives directly.

☐ 單變數的情形 $\frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}y} = 1$; 多變數的情形: 矩陣相乘爲單位矩陣。

☐ 有時候坐標變換關係式很複雜 (隱函數), 不易求偏導數, 計算反矩陣比較快。

Example 5. Consider $z = f(x, y)$, where all the second partial derivatives of f are continuous. Let $x = r\cos\theta$ and $y = r\sin\theta$.

(a) Express $\frac{\partial^2 r}{\partial x^2}$ and $\frac{\partial^2 \theta}{\partial x^2}$ in terms of r and θ.

(b) Express f_{xx} in terms of $r, \theta, f_r, f_\theta, f_{rr}, f_{r\theta}$, and $f_{\theta\theta}$.

Solution.

7xafpX9Dsbo

更進一步地，這裡也用直角坐標與極坐標的轉換關係學習函數的二次偏微分的鏈鎖律。養成好習慣，先把結構寫出來，再一個一個重問鏈鎖率，這樣比較不會漏寫或出錯。

Implicit Differentiation (隱函數微分)

The Chain Rule can be used to give a more complete description of the process of implicit differentiation. Suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x, that is, $y = y(x)$, where $F(x, y(x)) = 0$ for all x in the domain of y. If F is differentiable, we can apply the Chain Rule to differentiate both side of the equation $F(x, y(x)) = 0$ with respect to x to get

鏈鎖律的另一個應用是隱函數微分。直接從隱函數的關係式寫出兩變量之間的變化關係。

$$\Rightarrow$$

Implicit Function Theorem (page 942)**.** *If $F(x, y)$ is defined on a disk containing (x_0, y_0), where $F(x_0, y_0) = 0, F_y(x_0, y_0) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a function of x near the point (x_0, y_0) and the derivative of this function is $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}$.*

Exercise (page 946)**.** If $f(x, y) = 0$ define y as a function of x, show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2}{f_y^3}.$$

Now we suppose that z is given implicitly as a function $z = z(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F(x, y, z(x, y)) = 0$ for all (x, y) in the domain of z. If F and z are differentiable, then we can use the Chain Rule to differentiable the equation $F(x, y, z(x, y)) = 0$ as follows:

If $\frac{\partial F}{\partial z} \neq 0$, we solve $\frac{\partial z}{\partial x}$ and obtain

$$\frac{\partial z}{\partial x} = \qquad\qquad \frac{\partial z}{\partial y} =$$

VqBMCTRPhPA

隱函數定理在分析學中是一個非常重要的定理，在隱函數的式子當中，一些變數之間的關係隱諱不明，這個定理的重點是告知在什麼情況下，某個變數可以完全用其它變數的函數形式表達，也就是可以清楚地知道哪些可以設定成獨立的變數，哪些是依賴變數。

Implicit Function Theorem (page 943)**.** *If $F(x, y)$ is defined within a sphere containing (x_0, y_0, z_0), where $F(x_0, y_0) = 0, F_z(x_0, y_0, z_0) \neq 0$, and F_x, F_y and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (x_0, y_0, z_0) and the partial derivatives of this function are $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$.*

Example 6 (page 943)**.** Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1 = 0$.

Solution.

14.6 Directional Derivatives and the Gradient Vector, page 946

Directional Derivatives, page 946

Definition 1 (page 947)**.** The *directional derivative* (方向導數) of $f(x, y)$ at (x_0, y_0) in the direction of a *unit* vector $\mathbf{u} = (a, b)$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

ZFOzRYkaego

方向導數顧名思義就是去研究函數沿著某個方向的變化率，利用定義域上的單位向量指定方向，按照分量的比例分配增加的程度再算差商的極限。對應到的幾何意義如左圖所示：函數的圖形限制在通過 P 且包含指定向量的平面，研究這個函數圖形在 P 點的切線斜率。

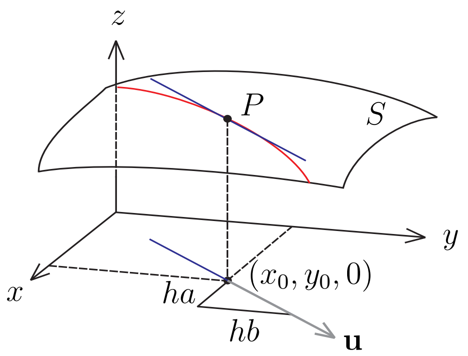

Figure 1: Directional derivative.

☐ $\mathbf{u}$ 必須是單位向量; 有時候只告知方向, 必須先把向量「單位化」後再計算方向導數。

Theorem 2 (page 948)**.** *If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = (a, b)$ and*

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b = (f_x(x, y), f_y(x, y)) \cdot (a, b).$$

可微分函數在一點沿著方向 $\mathbf{u}$ 的方向導數的算法，就是把函數對每個變數偏微分之後組成的向量與 $\mathbf{u}$ 向量內積。注意這裡的 $\mathbf{u}$ 必須是單位向量。

Proof. Define $g(h) = f(x(h), y(h)) = f(x_0 + ha, y_0 + hb)$, then by the definition of a directional derivative and the Chain Rule, we have

$$\begin{aligned} D_{\mathbf{u}} f(x_0, y_0) = g'(0) &= \left[\frac{\partial f}{\partial x} \frac{\mathrm{d}x}{\mathrm{d}h} + \frac{\partial f}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}h} \right] \bigg|_{h=0} \\ &= f_x(x_0, y_0)a + f_y(x_0, y_0)b = (f_x(x_0, y_0), f_y(x_0, y_0)) \cdot (a, b). \end{aligned}$$

☐

If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis, then we can write $\mathbf{u} = (\cos\theta, \sin\theta)$ and the directional derivative becomes

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta = (f_x(x, y), f_y(x, y)) \cdot (\cos\theta, \sin\theta).$$

The Gradient Vector, page 949

FbOc8tfJ014

函數對每個變數偏微分所組成的向量非常重要，我們把它取名爲梯度。所以前一頁的定理是說：函數的方向導數會是梯度與該方向的內積。

Definition 3 (page 950)**.** If f is a function of two variables x and y, then the *gradient* (梯度) of f is the vector function ∇f or grad f defined by

$$\nabla f(x, y) = \operatorname{grad} f(x, y) \overset{\text{def.}}{=} (f_x(x, y), f_y(x, y)) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Theorem 4 (page 950)**.** *If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = (a, b)$ and*

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

☐ 方向導數爲「梯度向量」與「單位向量」內積。

☐ 函數 $f(x, y)$ 的梯度向量 $\nabla f = (f_x, f_y)$ 是在 xy-平面上。

Maximizing the Directional Derivative, page 952

梯度向量的一個重要意義是它會指向函數增加最快的方向。這個性質由兩向量內積的幾何意義可以得知。

Theorem 5 (page 952)**.** *Suppose f is a differentiable function of two variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(x, y)$ is $|\nabla f(x, y)|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(x, y)$.*

Proof. Since $|\mathbf{u}| = 1$, we have

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos\theta = |\nabla f| \cos\theta,$$

where θ is the angle between ∇f and $\mathbf{u}$. The maximum value of $\cos\theta$ is 1 and this occurs when $\theta = 0$. Therefore the maximum value of $D_{\mathbf{u}} f$ is ∇f when $\mathbf{u}$ is the same direction as ∇f. ☐

☐ 柯西不等式 (Cauchy inequality) $\mathbf{u} \cdot \mathbf{v} \leq \|\mathbf{u}\|\|\mathbf{v}\|$。

Example 6. Let $f(x, y) = 2x^2 - xy + y^2 - 2x + y$.

(a) Find the directional derivative $D_{\mathbf{u}} f(p)$, where $p = (0, 0)$ and $\mathbf{u} = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

(b) Find the unit vector $\mathbf{v}$ that the directional derivative $D_{\mathbf{v}} f(p)$ is maximal.

Solution.

Functions of Three Variables, page 950

0-toVDDT2fU

Using vector notation, we can write the directional derivative in the compact form:

$$D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h} = \nabla f(\mathbf{x}_0) \cdot \mathbf{u}.$$

where $\mathbf{x}_0 = (x_0, y_0)$ if $n = 2$ and $\mathbf{x}_0 = (x_0, y_0, z_0)$ if $n = 3$.

Tangent Planes to Level Surfaces, page 954

Suppose S is a level surface with equation $F(x, y, z) = k$, and let $P(x_0, y_0, z_0)$ be a point on S. Let C be any curve that lies on the surface S and passes through the point P, that is, C is parameterized by $\mathbf{r}(t) = (x(t), y(t), z(t))$ and $\mathbf{r}(t_0) = (x(t_0), y(t_0), z(t_0)) = (x_0, y_0, z_0)$. Since C lies on S, we know

梯度向量的另一個重要的幾何意義是：梯度向量與等位面（或等高線）垂直。

$$F(x(t), y(t), z(t)) = k. \tag{3}$$

If x, y, and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate both sides of equation (3) as follows:

$$\frac{\partial F}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial F}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = 0 \Rightarrow \nabla F \cdot \mathbf{r}'(t) = 0.$$

In particular, when $t = t_0$, we have $\nabla F(x_0.y_0, z_0) \cdot \mathbf{r}'(t_0) = 0$

> The gradient vector at P, $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector $\mathbf{r}'(0)$ to any curve C on S that passes through P.

Definition 7 (page 954)**.** If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, it is therefore natural to define the *tangent plane to the level surface* $F(x, y, z) = k$ at P (等位面的切平面) as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. The equation is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0. \tag{4}$$

☐ 計算等位面的切平面, 梯度即爲切平面的法向量。

Definition 8 (page 954)**.** The *normal line* (法線) to S at P is the line passing through P and perpendicular to the tangent plane. The direction of the normal line is the gradient vector $\nabla F(x_0, y_0, z_0)$, and so its symmetric equations are

既然梯度與等位面垂直，那麼梯度就會是切平面的法向量，也可以當成是法線的方向向量。

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}. \tag{5}$$

☐ 若曲面可表示爲函數的圖形 $z = f(x, y)$, 可想成 $F(x, y, z) = z - f(x, y) = 0$。

Example 9 (page 941)**.** Find the equations of the tangent plane and normal line at $P(-2, 1, 3)$ to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$.

Solution.

Significance of the Gradient Vector, page 955

Vs_Rhm-QaKs

這個地方就再次統整梯度的幾何意義，對二變數函數來說，它的等高線(畫在定義域上)是一條曲線，而梯度向量（在定義域的平面上）與曲線處處垂直。對三變數函數來說，定義域是三度空間中的一個區域，函數的等位面是一個曲面，而梯度向量與曲面處處垂直。

Consider a function of two variables $f(x, y)$.

(1) The gradient vector ∇f is orthogonal to the level curve $f(x, y) = k$.

(2) The gradient vector ∇f gives the direction of fastest increases of f.

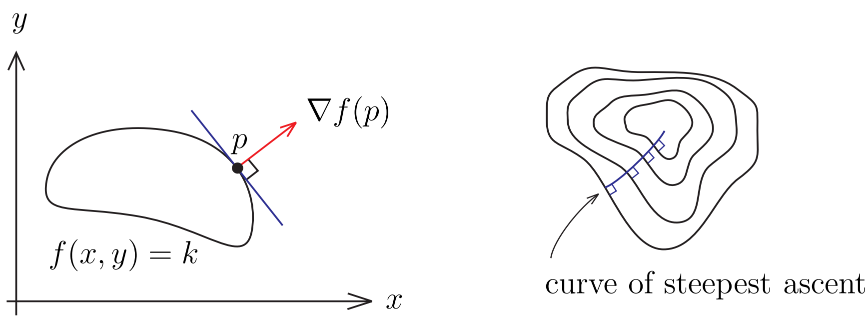

Figure 2: The gradient vector is orthogonal to the level curve.

Consider a function of three variables $F(x, y, z)$.

(1) The gradient vector ∇F is orthogonal to the level surface $F(x, y, z) = k$.

(2) The gradient vector ∇F gives the direction of fastest increases of F.

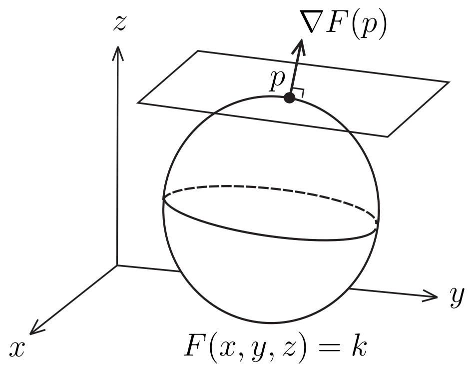

Figure 3: The gradient vector is orthogonal to the level surface.

Intersection of Two Surfaces

Suppose S_1 and S_2 are two surfaces determined by two equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$, respectively. The intersection of two surfaces is a space curve called C. Suppose that $\mathbf{r}(t)$ is a parametric equation of the space curve C and $\mathbf{r}(t_0) = P$, then $\mathbf{r}'(t_0)$ is parallel to $\nabla F(p) \times \nabla G(p)$.

kUFrgcN_mTA

回想高中所學兩個空間中的平面的交線之方向向量，計算方式就是直接把兩個平面的法向量外積而得；這裡也是一樣，兩個曲面的交集若交出一條空間曲線，這條曲線的切線找法，也可以直接把曲面各自的梯度向量外積而得。

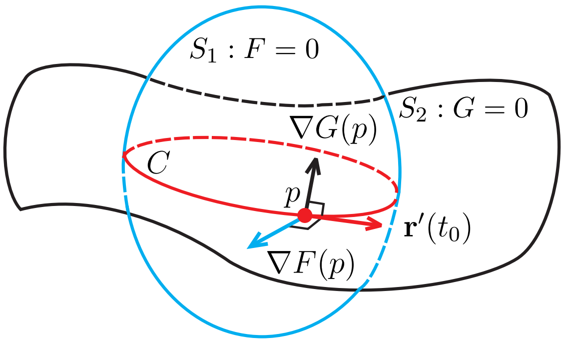

Figure 4: Intersection of two surfaces.

Example 10. Find the parametric equation of the tangent line to the curve of intersection of the surfaces $x^2 + 2y^2 + z^2 = 4$ and $x^2 + y^2 - z^2 = 1$ at the point $(1, 1, 1)$.

Solution.

14.7 Maximum and Minimum Values, page 959

uS8c920CeQg

和單變數函數的情形類似, 極值的意思也分成兩種情況: 局部極值與絕對極值。局部極值是存在一個鄰域, 在鄰域內的函數值與討論的函數值有大小關係; 而絕對極值是先指定一個範圍, 然後在那個範圍內的所有函數值與討論的函數值有大小關係。

Definition 1 (page 960)**.** A function of two variables has a *local maximum* (局部極大值處) at (x_0, y_0) if $f(x, y) \le f(x_0, y_0)$ when (x, y) is near (x_0, y_0). (This means that $f(x, y) \le f(x_0, y_0)$ for all points (x, y) in some disk with center (x_0, y_0).) The number $f(x_0, y_0)$ is called a *local maximum value* (局部極大值). If $f(x, y) \ge f(x_0, y_0)$ when (x, y) is near (x_0, y_0), then f has a *local minimum* (局部極小值處) at (x_0, y_0) and $f(x_0, y_0)$ is a *local minimum value* (局部極小值).

Definition 2 (page 960)**.** If the inequalities in Definition 1 hold for *all* points (x, y) in the domain of f, then f has an *absolute maximum* (最大值) or *absolute minimum* (最小值) at (x_0, y_0).

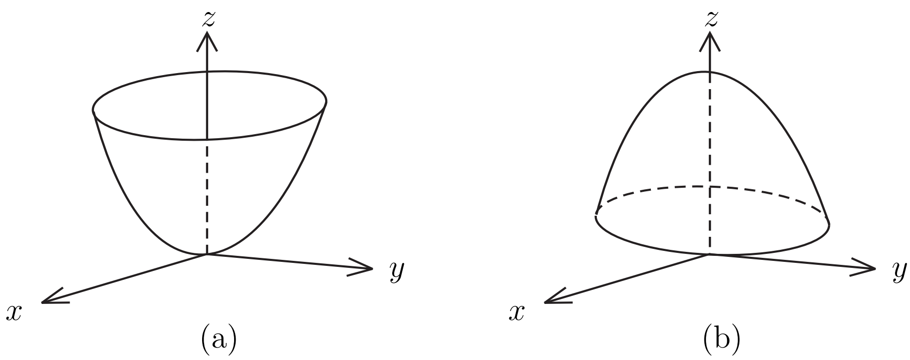

Figure 1: (a) Local and absolute minimum. (b) Local and absolute maximum.

定理說明局部極值的必要條件是函數在該點的所有偏導數都是零。這個定理雖然不是充份條件, 但它提供了找尋極值點以進行極值分類的資訊。

Theorem 3 (page 960)**.** *If f has a local maximum or minimum at (x_0, y_0) and the first-order partial derivatives of f exist there, then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.*

Proof. Let $g(x) = f(x, y_0)$. If f has a local maximum (or minimum) at (x_0, y_0), then $g(x)$ has a local maximum (or minimum) at x_0, so by Fermat's Theorem, we get $g'(x_0) = 0 = f_x(x_0, y_0)$. Similarly, by applying Fermat's Theorem to the function $\tilde{g}(y) = g(x_0, y_0)$, we obtain $g'(y_0) = 0 = f_y(x_0, y_0)$. □

□ 若函數在 (x_0, y_0) 是局部極值, 則 $\nabla f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0)) = (0, 0) = \mathbf{0}$ (零向量)。

這裡要做的事情和單變數函數的情況類似, 也是先定義臨界點, 然後針對臨界點討論極值問題。

Definition 4 (page 960)**.** A point (x_0, y_0) is called a *critical point* (臨界點) or *stationary point* (平穩點、駐點) of f if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, or if one of these partial derivatives does not exist.

□ 臨界點除了滿足 $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ 的點外, 還包括所有偏導數不存在的點。

□ 臨界點只是函數達到極值的必要條件, 非充分條件。

kBxx07emlqU

Example 5 (page 960)**.** Consider $f(x,y) = x^2 + y^2 - 2x - 6y + 14$, then

$$f_x(x,y) = 2x - 2 \qquad f_y(x,y) = 2y - 6.$$

These partial derivative are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is $(1,3)$. Since $f(x,y) = 4 + (x-1)^2 + (y-3)^3 \geq 4$ for all x and y, $f(1,3) = 4$ is a local minimum, and in fact it is the absolute minimum of f.

The graph of f is the ____________________ with vertex $(1,3,4)$.

先從二次函數介紹兩種臨界點的長相，其中一個會產生局部極小值（若函數全部加上負號則變成局部極大值），另一個會形成鞍點。

Example 6 (page 960)**.** Consider the function $f(x,y) = y^2 - x^2$. Since $f_x = -2x$ and $f_y = 2y$, the only critical point is ______. For points on the x-axis ($x \neq 0$) we have $f(x,0) = -x^2 < 0$ and for points on the y-axis ($y \neq 0$) we have $f(x,0) = y^2 > 0$. Thus every disk with center $(0,0)$ contains points where f takes positive values and negative values. Therefore, f has no extreme value.

The graph of f is the ________________________.

Definition 7 (page 961)**.** The graph of $z = y^2 - x^2$ has a horizontal tangent plane $z = 0$ at the origin. $f(0,0) = 0$ is a maximum in the direction of x-axis but a minimum in the direction of the y-axis. Near the origin the graph has the shape of a saddle and so $(0,0)$ is called a *saddle point* (鞍點) of f.

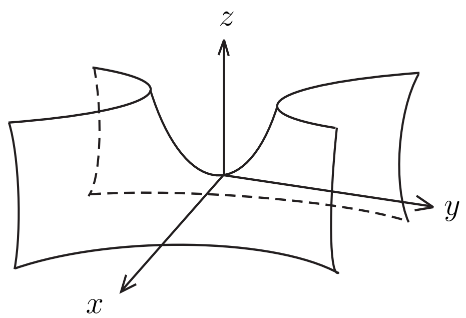

Figure 2: $(0,0)$ is a saddle point of $f(x,y) = y^2 - x^2$.

For a function of one variable $f(x)$, we use second derivative of $f(x)$ to detect the critical points are local maximum or local minimum. Here we will introduce the Second Derivative Test for functions of two variables to investigate the properties of critical points.

Definition 8. The *Hessian matrix* or *Hessian* (赫氏矩陣) of $f(x,y)$ at (x_0,y_0) is

$$\text{Hess}\,(f)(x_0,y_0) = \begin{bmatrix} f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0) \\ f_{yx}(x_0,y_0) & f_{yy}(x_0,y_0) \end{bmatrix}.$$

回想單變數函數的極值分類，必須從函數二次微分的符號判斷之；二變數函數的二次微分依變數個數及微分的先後順序不同而有四種可能，將它寫成矩陣的樣子稱爲赫氏矩陣。以下介紹的判別法會根據赫氏矩陣的行列式符號而下結論。

p1S9h9OCWUo

基本上赫氏矩陣的行列式符號代表函數凹口最大與最小的乘積 (正號代表凹口朝上、負號代表凹口朝下), 所以定理的結論可用推理的方式得到, 比方說 (a) 的條件知道凹口最大的數與凹口最小的數相乘爲正, 所以同正或同負, 而 $f_{xx} > 0$ 告知至少一數爲正, 因此另一數也是正的, 於是凹口朝上的臨界點會產生局部極小值。其它情況也試著自行推理。

Second Derivative Test (page 961)**.** *Suppose the second partial derivatives of f are continuous on a disk with center (x_0, y_0), and suppose that $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ (that is, (x_0, y_0) is a critical point of f). Let*

$$D(x_0, y_0) = \det(\text{Hess}(f)(x_0, y_0)) = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2.$$

(a) *If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then $f(x_0, y_0)$ is a local minimum.*

(b) *If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then $f(x_0, y_0)$ is a local maximum.*

(c) *If $D(x_0, y_0) < 0$, then $f(x_0, y_0)$ is not a local maximum or minimum.*

☐ 情況 (c) 是鞍點。

☐ 若 $D(x_0, y_0) = 0$, 三種情況 (極大、極小、鞍點) 都有可能發生; 必須用別的方法判斷。

Remark 9.

(1) 對稱矩陣可以正交對角化, 所以 $\text{Hess}(f) = PDP^{-1}$, 其中 D 是對角化矩陣; P 是坐標變換矩陣。

(2) 因爲 $\det(AB) = \det(BA)$, 所以行列式在坐標變換下不變; 即 $\det(\text{Hess}(f)) = \det(PDP^{-1}) = \det(PP^{-1}D) = \det(ID) = \det(D)$。

(3) $f_{xx} > 0$ 沿 x 方向凹口向上, 則對角矩陣其中一個值 (特徵值、固有值) 爲正; $f_{xx} < 0$ 沿 x 方向凹口向下, 則對角矩陣其中一個值 (特徵值、固有值) 爲負。

2w0-IdJKnGA

例題示範如何將函數的臨界點確實分類。用二次微分判別法確定臨界點的屬性。

Example 10. Find the extreme value (local maximum and minimum values and saddle points) of the function $f(x, y) = 2x^3 - 4xy + 3y^2$.

Solution.

Absolute Maximum and Minimum Values, page 965

Recall that for one variable function $f(x)$, the Extreme Value Theorem says that if f is continuous on a *closed* interval $[a, b]$, then f has an absolute maximum value and an absolute minimum value. Absolute maximum and absolute minimum points are happened at the critical points or endpoints.

aXDPv2LQqEQ

另一種極值問題是要找絕對極值。在介紹二變數函數的絕對極值定理前,必須先了解什麼是有界的閉集合。

We will introduce the Extreme Value Theorem of two variables in this section.

Definition 11 (page 965)**.**

(a) A *boundary point* (邊界點) of a set $D \subset \mathbb{R}^2$ is a point (x_0, y_0) such that every disk with center (x_0, y_0) contains points in D and also points not in D.

(b) A *closed set* (閉集) D in $\mathbb{R}^2$ is one that contains all its boundary points.

(c) A *bounded set* (有界集) D in $\mathbb{R}^2$ is one that is contained within some disk.

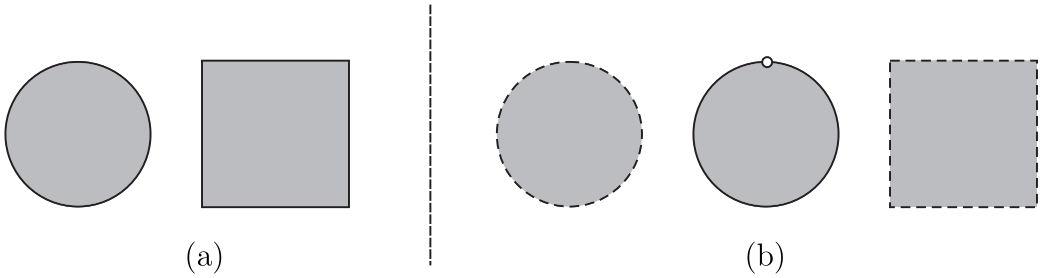

Figure 3: (a) Closed sets. (b) Sets that are not closed.

Extreme Value Theorem for Functions of Two Variables (page 965)**.** *If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.*

一個連續函數,當定義域是封閉且有界時,函數會有最大值與最小值。這個定理會在高等微積分課程詳細證明,而且到後期學到抽象化的概念下,封閉且有界的條件會轉變成緊緻集 (compact),而且歐氏空間的概念也可以抽象化變成討論測距空間 (metric space)。

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

(a) Find the values of f at the critical points of f in D.

(b) Find the extreme values of f on the boundary of D.

(c) The largest of the values from step 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

☐ 找「有界區域」的函數極值, 除了臨界點外, 邊界點也要納入考慮。

loeZzQiHXNQ

這個例題討論的區域是封閉而且有界，而且函數連續，所以絕對極值必存在。在區域內部，臨界點會是所有產生極值的候選點。至於邊界，這個例子是三個線段組成，所以分別將線段改寫，則函數限制在線段上變成單變數函數，再尋求極值。最後再把所有的候選點與邊界的極值點一起比大小。

Example 12. Find the extreme values of $f(x, y) = x^2 + xy + y^2 - 4x + 3y$ in the region bounded by $x = 0, y = 0$, and $x + y = 4$.

Solution.

5EmrV5_jyH0

這個例題也是在問函數的絕對極值，因爲區域的邊界是圓形，所以也可以很容易地將邊界參數化，所以函數限制在邊界上也形成單變數函數，這樣就可以了解邊界上函數的極值。若區域的邊界更複雜的話，下一個單元會介紹拉格朗日乘子法，可用那個方法處理。

Example 13. Find the absolute maximum and minimum values of $f(x, y) = 4x + 6y - x^2 - y^2$ in the region $x^2 + y^2 \leq 1$.

Solution.

Appendix, page 967

Proof of the Second Derivative Test. We compute the second-order directional derivative of f in the direction of $\mathbf{u} = (h, k)$. The first-order derivative is

$$D_{\mathbf{u}} f = f_x h + f_y k.$$

Apply this theorem a second time, we have

$$\begin{aligned} D^2_{\mathbf{u}} f = D_{\mathbf{u}}(D_{\mathbf{u}} f) = \nabla(D_{\mathbf{u}} f) \cdot \mathbf{u} &= \left(\frac{\partial}{\partial x}(f_x h + f_y k), \frac{\partial}{\partial y}(f_x h + f_y k) \right) \cdot (h, k) \\ &= (f_{xx} h + f_{yx} k)h + (f_{xy} h + f_{yy})k = f_{xx} h^2 + 2 f_{xy} hk + f_{yy} k^2 \\ &= f_{xx} \left(\frac{f_{xx} h + f_{xy} k}{f_{xx}} \right)^2 + \frac{k^2}{f_{xx}} (f_{xx} f_{yy} - f_{xy}^2). \end{aligned}$$

Remark that $D(x_0, y_0) = \det(\text{Hess}(f)(x_0, y_0)) = \left(f_{xx} f_{yy} - f_{xy}^2 \right)\big|_{(x_0, y_0)}$.

(a) If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then $D^2_{\mathbf{u}} f(x_0, y_0) \geq 0$ for any direction $\mathbf{u}$. If $D^2_{\mathbf{u}} f(x_0, y_0) = 0$ for some direction $\mathbf{u}$, then $k = 0$ and $f_{xx} h + f_{xy} k = 0$. However, it implies $(h, k) = (0, 0)$ and it contradicts to $h^2 + k^2 = 1$. Hence $f(x_0, y_0)$ is a local minimum.

(b) If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then $D^2_{\mathbf{u}} f(x_0, y_0) \leq 0$ for any direction $\mathbf{u}$. If $D^2_{\mathbf{u}} f(x_0, y_0) = 0$ for some direction $\mathbf{u}$, then $k = 0$ and $f_{xx} h + f_{xy} k = 0$. However, it implies $(h, k) = (0, 0)$ and it contradicts to $h^2 + k^2 = 1$. Hence $f(x_0, y_0)$ is a local maximum.

(c) If $D(x_0, y_0) < 0$, we will find two different directions such that the signs of the second derivatives of f are different.

⋆ If $f_{xx} > 0$ and $f_{yy} < 0$, then choosing $\mathbf{u} = (h, k) = (1, 0)$ implies $D^2_{\mathbf{u}} f = f_{xx} > 0$. If we choose $(h, k) = (0, 1)$, then $D^2_{\mathbf{u}} f = f_{yy} < 0$.

⋆ If $f_{xx} < 0$ and $f_{yy} > 0$, then choosing $\mathbf{u} = (h, k) = (1, 0)$ implies $D^2_{\mathbf{u}} f = f_{xx} < 0$. If we choose $(h, k) = (0, 1)$, then $D^2_{\mathbf{u}} f = f_{yy} > 0$.

⋆ If $f_{xx} > 0$ and $f_{yy} > 0$, then choosing $\mathbf{u} = (h, k) = (1, 0)$ implies $D^2_{\mathbf{u}} f = f_{xx} > 0$. If we choose $(h, k) /\!/ (f_{xy}, -f_{xx})$ such that $k \neq 0$, then $D^2_{\mathbf{u}} f = \left(\frac{f_{xx} f_{yy} - f_{xy}^2}{f_{xx}} \right) k^2 < 0$.

⋆ If $f_{xx} < 0$ and $f_{yy} < 0$, then choosing $\mathbf{u} = (h, k) = (1, 0)$ implies $D^2_{\mathbf{u}} f = f_{xx} < 0$. If we choose $(h, k) /\!/ (f_{xy}, -f_{xx})$ such that $k \neq 0$, then $D^2_{\mathbf{u}} f = \left(\frac{f_{xx} f_{yy} - f_{xy}^2}{f_{xx}} \right) k^2 > 0$.

Hence $f(x_0, y_0)$ is not a local maximum or minimum.

□

X2_8vJN8LyA

附錄提供二次微分判別法有關局部極值定理的證明，這裡是直接計算函數沿著指定的方向 $\mathbf{u} = (h, k)$ 的二次微分，得到對於 h, k 的二次式，利用配方法根據二次式的符號（正好對應於定理的條件）得到局部極值的結論。

14.8 Lagrange Multipliers, page 971

Q_ZJQm6btWo

拉格朗日乘子法是在處理限制條件下的極值問題， 前一節討論的極值問題，因爲變數之間彼此獨立，所以直接對各個變數偏微找出臨界點而尋求答案。現在因爲變數之間必須滿足一個關係式，所以若用之前的方法不加以修改，臨界點不見得會滿足這個限制條件。由此可以再重新想一想拉格朗日乘子法是如何造出新的目標函數以滿足所有條件。

Method of Lagrange Multipliers (page 972)**.** *To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$):*

(a) *Find all values of x, y, z, and λ such that*

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \qquad \text{and} \qquad g(x, y, z) = k.$$

(b) *Evaluate f at all the points (x, y, z) that result from step (a). The largest (smallest) of these values is the maximum (minimum) value of f.*

Proof. For $\lambda \in \mathbb{R}$, consider $F(x, y, z, \lambda) = f(x, y, z) - \lambda(g(x, y, z) - k)$, then on the level set $g(x, y, z) = k$, the function $F(x, y, z, \lambda)$ and $f(x, y, z)$ take the same value. So to find the extreme value of $f(x, y, z)$ on the surface $g(x, y, z) = k$ is equivalent to find the extreme value of $F(x, y, z, \lambda)$, and it implies

$$\nabla F(x, y, z, \lambda) = \mathbf{0} \quad \Leftrightarrow \quad \nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = k.$$

□

ul1c1LbqXtU

這裡先用各位在高中時就會用空間幾何的方法找出平面外一點到平面的最短距離的例子，現在試著用拉格朗日乘子的方法求解，目標與操作比較明確；高中的解法某種程度來說比較流於形式，也只適用於點到平面的情況。拉格朗日乘子法適用於點到曲面的情形。

Example 1 (page 964)**.** Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

Solution.

□ 和高中時所學到的解法 (利用距離最短的幾何性質) 相對照。

□ 此問題可以把變數 z 用 $4 - x - 2y$ 換掉, 然後換成二變數函數的極值問題。

Example 2 (page 964, 973)**.** A rectangular box without a lid is to be made from $12\,\mathrm{m}^2$ of a cardboard. Find the maximum volume of such a box.

quJJXhw9rp0

製造無蓋紙盒有最大的體積問題也可以用拉格朗日乘子法處理，因爲紙盒的表面積有限制。各位也可以想一想這個問題如何用算幾不等式處理。

Solution.

Example 3. Find the maximum and minimum values of the function $f(x, y) = x^3 + 3x^2y$ in the region $x^2 + 4xy + 5y^2 \leq 5$.

zlbJk1dE54E

按照連續函數的極值定理：有界封閉區域上的連續函數必有最大最小值。實作層面，對於區域內部直接找臨界點，而邊界的極值就可以用拉格朗日乘子法處理。最後再把所有的臨界點與邊界的極值點一起比大小。

Solution.

Two Constraints, page 976

nxtcsb0gCPw

拉格朗日乘子法可用於多個限制條件下的極值問題，若有三個變數的函數在兩個限制條件下的極值問題，每一個限制條件就再創一個變數，得到新的目標函數爲五變數函數，因爲在限制條件下新的目標函數與原目標函數取值一樣，所以改研究新的目標函數之極值，而新的目標函數對於新創的兩個變數偏微令成零之下即爲限制條件。

We want to find the maximum and minimum values of a function $f(x, y, z)$ subject to two constraints of the form $g(x, y, z) = k$ and $h(x, y, z) = c$. Let

$$F(x, y, z, \lambda, \mu) = f(x, y, z) - \lambda(g(x, y, z) - k) - \mu(h(x, y, z) - c).$$

For $\lambda, \mu \in \mathbb{R}$, under these two constraints, $F(x, y, z, \lambda, \mu)$ and $f(x, y, z)$ take the same value. So to find the extreme values of $f(x, y, z)$ with $g(x, y, z) = k$ and $h(x, y, z) = c$ is equivalent to find the extreme values of $F(x, y, z, \lambda, \mu)$. That is,

$$\nabla F = \mathbf{0} \quad \Leftrightarrow \quad \nabla f = \lambda \nabla g + \mu \nabla h \quad \text{and} \quad g(x, y, z) = k \quad \text{and} \quad h(x, y, z) = c.$$

Example 4. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

Solution.

Example 5 (page 977)**.** Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

x9SmS_h5aS8

拉格朗日乘子法的概念容易，但是實作層面上的難點是聯立方程式的求解困難，如何適當地分情況討論方程式的解是一門學問。若遇到更複雜的式子求解那麼必須借助電腦的幫忙。

Solution.