Chapter 13 Vector Functions

13.1 Vector Functions and Space Curves (page 848)

We now study functions whose values are vectors because such functions are needed to describe curves and surfaces in space.

Wwwa5xVCCrE

這一章本身有一個主體性，是在了解空間曲線理論；另一方面，前兩節的向量函數概念也是之後雙變數函數理論的前置作業。簡單說來，之前都是在探討單變數函數，函數圖形也是放在 xy 平面上觀察，從參數式的觀點下，現在要對於「維度」開始推廣，變成現在所要討論的向量函數。

Definition 1 (page 848)**.** A *vector-valued function*, or *vector function* (向量函數), is a function whose domain is a set of real numbers and whose range is a set of vectors.

Here we will focus on vector functions $\mathbf{r}$ whose values are three-dimensional vectors. This means that for every number t in the domain of $\mathbf{r}$ there is a unique vector in $\mathbb{R}^3$ denoted by $\mathbf{r}(t)$. We can write

$$\mathbf{r}(t) = (f(t), g(t), h(t)) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k},$$

where f, g, h are real-valued functions of t called the *component functions* (分量函數) of $\mathbf{r}$.

☐ 定義中 $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, $\mathbf{k} = (0,0,1)$ 爲 $\mathbb{R}^3$ 中的直角坐標向量。

☐ 課本用尖括號 $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ 表示向量函數, 但大部分文獻仍使用小括號。

Example 2 (page 848)**.** If $\mathbf{r}(t) = (t^3, \ln(3-t), \sqrt{t})$, then the component functions are

$$f(t) = t^3, \qquad g(t) = \ln(3-t), \quad \text{and} \quad h(t) = \sqrt{t}.$$

By the usual convention, the *domain* (定義域) of $\mathbf{r}$ consists of all values of t for which the expresstion for $\mathbf{r}(t)$ is defined. Therefore the domain of $\mathbf{r}$ is $[0, 3)$.

Definition 3 (page 848)**.** The *limit* of a vector function $\mathbf{r}$ is defined by taking the limits of its components functions as follows. If $\mathbf{r}(t) = (f(t), g(t), h(t))$, then

$$\lim_{t\to a} \mathbf{r}(t) = \Big(\lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\Big) = \lim_{t\to a} f(t)\,\mathbf{i} + \lim_{t\to a} g(t)\,\mathbf{j} + \lim_{t\to a} h(t)\,\mathbf{k}$$

provided the limits of the component functions exist.

Definition 4 (page 849)**.** A vector function $\mathbf{r}$ is *continuous at a* if $\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$.

☐「連續」可以看成「函數」與「極限」可以互換。

☐ 向量函數 $\mathbf{r}$ (在 $t = a$) 連續若且唯若所有分量函數 $f(t), g(t), h(t)$ (在 $t = a$) 連續。

There is a closed connection between vector-valued functions and space curves.

Definition 5 (page 849)**.** The <u>set</u> C of all points (x, y, z) in space, where

$$x = f(t), \qquad y = g(t), \qquad z = h(t), \tag{1}$$

gnA8LWTfFFY

研究空間曲線的方式是用參數方程理解它，這樣就可以用微積分的理論了解曲線的幾何性質。

and t varies throughout the interval I, is called a *space curve* (空間曲線). The equations in (1) are called *parametric equations of C* (參數方程) and t is called a *parameter* (參數).

☐ 有時候 $\mathbf{r}(t)$ 也稱爲位置向量 (position vector)。

☐「空間曲線」是 $\mathbb{R}^3$ 中的一些點所成的「集合」, 可能有很多不同的「參數方程」表達。

螺旋線算是空間曲線論中最標準的一個模型，我們會用螺旋線認識各式幾何量，再深入探討其理論。

Example 6 (page 849)**.** The curve $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ is called a *helix* (螺旋線).

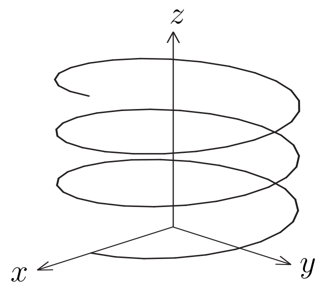

Figure 1: A helix $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}, 0 \le t \le 6\pi$.

現在要學習如何把曲線的參數方程表達；特別是有些曲線是兩個曲面的交集，該如何寫出其參數方程是需要一些經驗的累積。

Example 7.

(a) Find a vector equation and parametric equations for the line that join the point $A(a_1, a_2, a_3)$ to the point $B(b_1, b_2, b_3)$.

(b) Find a vector equation and parametric equations that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

Solution.

Using computers to draw space curves, page 851

現在有非常多的數學繪圖軟體，只要輸入參數方程，它就可以在電腦上呈現曲線的長相，可幫助我們更快認識空間曲線。

(a) Toroidal spiral: $x = (4 + \sin 7t)\cos t,\ y = (4 + \sin 7t)\sin t,\ z = \cos 7t,\ 0 \le t \le 2\pi$.

(b) Trefoil knot: $x = (2 + \cos 1.5t)\cos t,\ y = (2 + \cos 1.5t)\sin t,\ z = \sin 1.5t,\ 0 \le t \le 4\pi$.

(c) Twisted cubic: $x = t, \quad y = t^2, \quad z = t^3$.

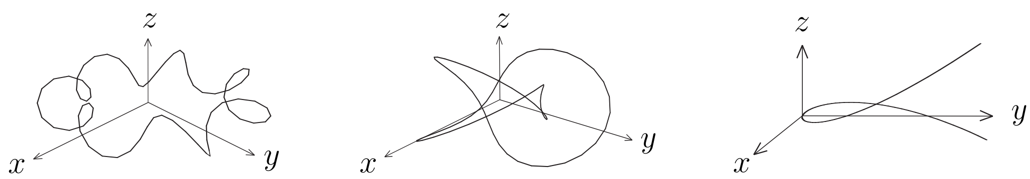

Figure 2: (a) Toroidal spiral. (b) Trefoil knot. (c) Twisted cubic.

Exercise (page 855)**.** Find a vector function that represents the curve of intersection of the two surfaces.

(a) The cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1 + y$.

(b) The semiellipsoid $x^2 + y^2 + 4z^2 = 4, y \ge 0$, and the cylinder $x^2 + z^2 = 1$.

13.2 Derivatives and Integrals of Vector Functions (page 855)

Derivatives, page 855

Definition 1 (page 855)**.** The *derivative* (導函數) $\mathbf{r}'(t)$ of a vector function $\mathbf{r}(t)$ is

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}.$$

qVOC-rk4i6I

這一節要認識一些向量函數的微積分理論，向量函數的導函數就是每個分量都各自算導函數，放在一起就形成一個向量，它會是曲線的切向量；只要切向量不是零向量，那麼就可以定義曲線的切線以及單位切向量。

Definition 2 (page 856)**.**

(a) The vector $\mathbf{r}'(t_0)$ is called the *tangent vector* (切向量) to the curve C defined by $\mathbf{r}(t)$ at the point $P = \mathbf{r}(t_0)$, provided that $\mathbf{r}'(t_0)$ exists and $\mathbf{r}'(t_0) \neq \mathbf{0}$.

(b) The *tangent line* (切線) to the curve C at $P = \mathbf{r}(t_0)$ is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t_0)$.

(c) The *unit tangent vector* (單位切向量) is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

Theorem 3 (page 856)**.** *If $\mathbf{r}(t) = (f(t), g(t), h(t)) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions, then*

$$\mathbf{r}'(t) = (f'(t), g'(t), h'(t)) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}.$$

Theorem 4 (page 858)**.** *Suppose $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are differentiable vector functions, c is a scalar, and $f(t)$ is a real-valued function. Then*

Ps6fqT_Cxsg

向量函數的四則運算與微分的規則，因爲向量之間又還有內積與外積的計算，所以又會多了一些關係式；在看定理前，必須先了解每個符號的本質：在運算之前與之後到底是數字、函數、還是向量必須確認清楚。

(1) $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t) + \mathbf{v}(t)) = \mathbf{u}'(t) + \mathbf{v}'(t)$.

(2) $\frac{\mathrm{d}}{\mathrm{d}t}(c\,\mathbf{u}(t)) = c\,\mathbf{u}'(t)$.

(3) $\frac{\mathrm{d}}{\mathrm{d}t}(f(t)\,\mathbf{u}(t)) = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)$.

(4) $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$.

(5) $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$.

(6) $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(f(t))) = \mathbf{u}'(f(t))f'(t)$.

Proof. Let $\mathbf{u}(t) = u_1(t)\,\mathbf{i} + u_2(t)\,\mathbf{j} + u_3(t)\,\mathbf{k}$ and $\mathbf{v}(t) = v_1(t)\,\mathbf{i} + v_2(t)\,\mathbf{j} + v_3(t)\,\mathbf{k}$.

(1)
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t) + \mathbf{v}(t)) &= \lim_{h\to 0} \frac{\mathbf{u}(t+h) + \mathbf{v}(t+h) - (\mathbf{u}(t) + \mathbf{v}(t))}{h} \\ &= \lim_{h\to 0} \frac{\mathbf{u}(t+h) - \mathbf{u}(t)}{h} + \lim_{h\to 0} \frac{\mathbf{v}(t+h) - \mathbf{v}(t)}{h} = \mathbf{u}'(t) + \mathbf{v}'(t).\end{aligned}$$

(2)
$$\frac{\mathrm{d}}{\mathrm{d}t}(c\,\mathbf{u}(t)) = \lim_{h\to 0}\frac{c\,\mathbf{u}(t+h) - c\,\mathbf{u}(t)}{h} = c\,\lim_{h\to 0}\frac{\mathbf{u}(t+h)-\mathbf{u}(t)}{h} = c\,\mathbf{u}'(t).$$

(4)
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t)\cdot\mathbf{v}(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}(u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t))\\ &= u_1'(t)v_1(t) + u_2'(t)v_2(t) + u_3'(t)v_3(t) + u_1(t)v_1'(t) + u_2(t)v_2'(t) + u_3(t)v_3'(t)\\ &= \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t).\end{aligned}$$

(6)
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(f(t))) &= \frac{\mathrm{d}}{\mathrm{d}t}(u_1(f(t))\,\mathbf{i} + u_2(f(t))\,\mathbf{j} + u_3(f(t))\,\mathbf{k})\\ &= u_1'(f(t))f'(t)\,\mathbf{i} + u_2'(f(t))f'(t)\,\mathbf{j} + u_3'(f(t))f'(t)\,\mathbf{k}\\ &= (u_1'(f(t))\,\mathbf{i} + u_2'(f(t))\,\mathbf{j} + u_3'(f(t))\,\mathbf{k})f'(t) = \mathbf{u}'(f(t))f'(t).\end{aligned}$$

□

Exercise. Show that

(3) $\frac{\mathrm{d}}{\mathrm{d}t}(f(t)\,\mathbf{u}(t)) = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)$.

(5) $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}(t)\times\mathbf{v}(t)) = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$.

xUFKBF-aRUk

Example 5 (page 858)**.** If $\|\mathbf{r}(t)\| = c$ (a constant), then $\mathbf{r}(t)\cdot\mathbf{r}(t) = c^2$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}(t)\cdot\mathbf{r}(t)) =$$

這個式子算式看起來簡單，但很容易被人忽略。我們可以把這件事情賦予幾何意義：若曲線落在半徑爲 c 的球上，球心與坐標中心一致，則曲線的位置向量與切向量處處垂直。

Thus $\mathbf{r}'(t)\cdot\mathbf{r}(t) = 0$, which says that ______________________________.

Exercise (page 861)**.** If $\mathbf{r}(t) \neq 0$, show that $\dfrac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{r}(t)\| = \dfrac{\mathbf{r}(t)\cdot\mathbf{r}'(t)}{\|\mathbf{r}(t)\|}$.

Integrals, page 859

向量函數的積分，基本上就是每個分量函數各自積分的意思。

The *definite integral* (定積分) of a continuous vector function $\mathbf{r}(t)$ is

$$\begin{aligned}\int_a^b \mathbf{r}(t)\,\mathrm{d}t &= \lim_{n\to\infty}\sum_{i=1}^n \mathbf{r}(t_i^*)\Delta t\\ &= \lim_{n\to\infty}\left(\left(\sum_{i=1}^n f(t_i^*)\Delta t\right)\mathbf{i} + \left(\sum_{i=1}^n g(t_i^*)\Delta t\right)\mathbf{j} + \left(\sum_{i=1}^n h(t_i^*)\Delta t\right)\mathbf{k}\right)\\ &= \left(\int_a^b f(t)\,\mathrm{d}t\right)\mathbf{i} + \left(\int_a^b g(t)\,\mathrm{d}t\right)\mathbf{j} + \left(\int_a^b h(t)\,\mathrm{d}t\right)\mathbf{k}.\end{aligned}$$

We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t)\,\mathrm{d}t = \Big[\mathbf{R}(t)\Big]\Big|_{t=a}^{t=b} = \mathbf{R}(b) - \mathbf{R}(a),$$

where $\mathbf{R}$ is an antiderivative of $\mathbf{r}$, that is, $\mathbf{R}'(t) = \mathbf{r}(t)$. We use the notation $\int \mathbf{r}(t)\,\mathrm{d}t$ for *indefinite integrals* (不定積分).