Chapter 11 Infinite Sequences and Series

11.1 Sequences (page 694)

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這一章的終極目標是泰勒展開式的理論，爲徹底了解泰勒展開式，必須先從數列開始，然後討論級數，最後介紹函數項級數。

這章用到非常多的數學論述與邏輯推演，必須反覆思考以逐漸體會。

Definition 1 (page 694)**.**

(1) A *sequence* (數列) is a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots.$$

The number a_1 is called the *first term* (第一項), a_2 is the *second term* (第二項), and in general a_n is the *n-th term* (第 n 項).

(2) An *infinite sequence* (無窮數列) is a sequence that each term a_n has a successor a_{n+1}.

(3) The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

☐ 微積分課程感興趣的是無窮數列，若將無窮數列依序寫下時，在一般項後面還會再加上「點點點」。

認識數列的幾種表示法。微積分課程中主要探討的數列有兩種類型，一種是有明確表達式的數列，另一種是利用遞迴式定義出的數列。

Example 2 (page 694)**.** Some sequences can be defined by giving a formula for the n-th term. There are three methods to describe a sequence. Notice that n doesn't have to start at 1.

(a) $\{\frac{n}{n+1}\}_{n=1}^{\infty}$, $a_n = \frac{n}{n+1}$, $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots\}$.

(b) $\{\frac{(-1)^n(n+1)}{3^n}\}_{n=1}^{\infty}$, $a_n = \frac{(-1)^n(n+1)}{3^n}$, $\{-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \ldots, \frac{(-1)^n(n+1)}{3^n}, \ldots\}$.

(c) $\{\sqrt{n-3}\}_{n=3}^{\infty}$, $a_n = \sqrt{n-3}, n \geq 3$, $\{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\}$.

(d) $\{\cos\frac{n\pi}{6}\}_{n=0}^{\infty}$, $a_n = \cos\frac{n\pi}{6}, n \geq 0$, $\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots, \cos\frac{n\pi}{6}, \ldots\}$.

☐ 數列不見得一定要從第一項開始寫起，可以從第三項或是第零項開始，例如 (c) 與 (d) 的說明。

Example 3 (page 695)**.** Here are some sequences that don't have a simple defining equation.

(a) The *Fibonacci sequence* (費波那契數列) $\{f_n\}$ is defined recursively by the conditions

$$f_1 = f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}, \quad n \geq 3.$$

The first few terms are $\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}$. This sequence arose when the 13th-century Italian mathematician known as Fibonacci solved a problem concerning the breeding of rabbits.

(b) If we let a_n be the digit in the n-th decimal place of the number $\sqrt{2}$, then $\{a_n\}$ is a well-defined sequence whose first few terms are $\{4, 1, 4, 2, 1, 3, 5, 6, 2, \ldots\}$.

☐ 微積分課程中主要是討論有一般式或是前後項有關係的數列，對於 (b) 會有別的數學理論處理。

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Definition 4 (page 696)**.** (數列極限之收斂或發散)

數列收斂意思是極限值存在 (實數), 否則稱發散。 而 $\lim\limits_{n\to\infty} a_n = \infty$ 的情形, 是發散的數列。

(1) A sequence $\{a_n\}$ has the *limit* L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \ \text{ as } \ n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

(2) If $\lim\limits_{n\to\infty} a_n$ exists, we say the sequence *converges* (or is *convergent*, 收斂). Otherwise, we say the sequence *diverges* (or is *divergent*, 發散).

(3) If a_n becomes large as n becomes large, we use the notation $\lim\limits_{n\to\infty} a_n = \infty$.

數列的極限相關定理與函數的極限雷同, 也有四則運算與夾擠定理。

Theorem 5. *If $\lim\limits_{n\to\infty} a_n$ exists, then it is unique.*

Property 6 (Limit Laws for Sequences, page 697)**.** *If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then*

(1) $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$.

(2) $\lim\limits_{n\to\infty} (a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$.

(3) $\lim\limits_{n\to\infty} c\, a_n = c \lim\limits_{n\to\infty} a_n$. *In particular,* $\lim\limits_{n\to\infty} c = c$.

(4) $\lim\limits_{n\to\infty} (a_n b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$.

(5) $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ *if* $\lim\limits_{n\to\infty} b_n \neq 0$.

(6) $\lim\limits_{n\to\infty} a_n^p = \left(\lim\limits_{n\to\infty} a_n\right)^p$ *if* $p > 0$ *and* $a_n > 0$.

The Squeeze Theorem (夾擠定理, page 698)**.** *If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.*

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Theorem 7. *If $\lim\limits_{n\to\infty} a_n = L$, then the limit of any subsequences $\lim\limits_{k\to\infty} a_{n_k} = L$.*

子數列必須依序挑選原數列的數字, 因爲子數列有保持順序, 所以收斂性也會被繼承。

☐ 極限若存在, 真相 (極限值) 只有一個!

☐ 數列的極限與函數的極限一樣有「四則運算」以及「夾擠定理」。

☐ 夾擠定理, 只要確定某一項之後三個數列有大小關係即可, 和前面有限項的大小無關。

☐ 子數列存在性定理一般的應用是考慮其否逆命題 — 證明原數列極限不存在。

當一個數列有正有負, 可以先不考慮符號分析極限值, 若極限爲零, 則原數列限極亦爲零。這個定理算是很常使用。

Theorem 8 (page 698)**.** *If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.*

Proof. Since _______________, by the _______________, we have $\lim\limits_{n\to\infty} a_n = 0$. ☐

☐ 數列加絕對值之後的極限必須是零, 原數列極限才是零。若是其它數字都沒有相應的結論。

Theorem 9 (page 697)**.** *If* $\lim_{x\to\infty} f(x) = L$ *and* $f(n) = a_n$ *when* n *is an integer, then* $\lim_{n\to\infty} a_n = L$.

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這兩個定理將函數的極限與數列的極限串聯起來，定理下方的三個註記也應好好體會。

Theorem 10 (page 699)**.** *If* $\lim_{n\to\infty} a_n = L$ *and the function* f *is continuous at* L*, then*

$$\lim_{n\to\infty} f(a_n) = f(L).$$

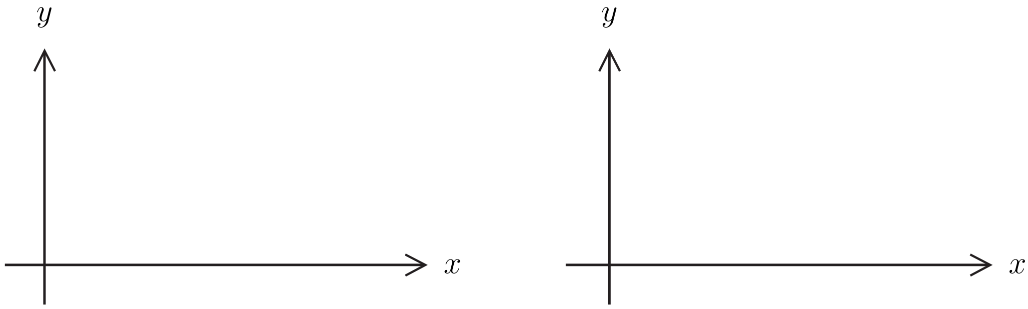

Figure 1: Limit relations between functions and sequences.

☐ 有了定理 9、定理 10, 就可以將上學期學過函數的極限運用到數列的極限, 超好用!

☐ 定理 10 意義:「連續函數」才可以和數列的「極限」交換順序。

☐ 若 $\lim_{n\to\infty} a_n = 0$, 則 $\lim_{n\to\infty} |a_n| = \left|\lim_{n\to\infty} a_n\right| = 0$。(因爲絕對值函數爲連續函數)

Example 11. Discuss the convergence or divergence of the following sequences:
(a) $a_n = \frac{-n^2+1}{2n^2+3n}$ (b) $b_n = \frac{n!}{n^n}$ (c) $c_n = \frac{(-1)^n}{n}$ (d) $d_n = \frac{\ln n}{n}$ (e) $e_n = \sin(\frac{\pi}{n})$.

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例題示範如何利用學到的定理處理數列的極限; 對於例題 (d) 應再強調的是: 因爲 n 是自然數, 並沒有離散型的羅必達法則, 所以若要使用羅必達法則, 必須過渡到改成變數爲實數 x 之後再使用。

Solution.

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關於等比數列，不論是定理的所有推論，還有它的結論都很重要，必須好好體會。

Theorem 12 (page 700)**.** *The sequence* $\{r^n\}_{n=1}^{\infty}$ *is convergent if* $-1 < r \le 1$ *and divergent for all other values of* r. *Furthermore, we have*

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1. \end{cases}$$

Proof. Consider $f(x) = a^x$. We know $\lim_{x\to\infty} a^x = \infty$ if $a > 1$; $\lim_{x\to\infty} a^x = 0$ if $0 < a < 1$.

(1) Put $\underline{a = r}$, we have

(2) If $\underline{r = 1}$,

(3) If $\underline{r = 0}$,

(4) If $\underline{-1 < r < 0}$,

(5) If $\underline{r = -1}$,

(6) If $\underline{r < -1}$,

□

Exercise. Show that $\lim_{n\to\infty} nr^n = 0$ if $|r| < 1$.

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認識遞增、遞減、單調數列與有界數列的意義。

Definition 13 (page 700)**.** A sequence $\{a_n\}$ is called *increasing* (遞增) if $a_n < a_{n+1}$ for all $n \ge 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called *decreasing* (遞減) if $a_n > a_{n+1}$ for all $n \ge 1$. A sequence is *monotonic* (單調) if it is either increasing or decreasing.

Definition 14 (page 701)**.** A sequence $\{a_n\}$ is *bounded above* (有上界) if there is a number M such that $a_n \le M$ for all $n \ge 1$. It is *bounded below* (有下界) if there is a number m such that $m \le a_n$ for all $n \ge 1$. If it is bounded above and below, then $\{a_n\}$ is a *bounded sequence* (有界數列).

Monotonic Sequence Theorem (page 702)**.** *Every bounded, monotonic sequence is convergent.* (單調有界數列必收斂。)

定理應拆解成兩句話理解：遞增有上界的數列收斂；遞減有下界的數列收斂。有界與單調兩條件缺一不可。

Figure 2: Monotonic sequence theorem.

- ☐ 有界數列未必收斂, 例如: __________________________。
- ☐ 單調數列未必收斂, 例如: __________________________。
- ☐ 定理證明要用到實數的完備性公設 (completeness axiom), 會在高等微積分的課程中詳細討論。

Example 15 (page 703)**.** Investigate the sequence $\{a_n\}_{n=1}^{\infty}$ defined by the *recurrence relation* (遞迴關係): $a_1 = 2, a_{n+1} = \frac{1}{2}(a_n + 6)$ for $n = 1, 2, 3, \ldots$.

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數學上必須先證明數列的極限存在, 最後一段的極限值找法才有意義。也就是說, 利用數學歸納法證明這個遞迴數列的極限存在的證明是必須的。

Solution. Monotone: We claim: $a_{n+1} > a_n$ for all $n \in \mathbb{N}$.

(1) When $n = 1$,

(2) Assume that it is true for $n = k$, that is, $a_{k+1} > a_k$.

(3) When $n = k + 1$,

(4) By ______________________, we know $\{a_n\}$ is monotone.

Bounded: We claim: $a_n < 6$ for all $n \in \mathbb{N}$.

(1) When $n = 1$,

(2) Assume that it is true for $n = k$, that is, $a_k < 6$.

(3) When $n = k + 1$,

(4) By ______________________, we know $\{a_n\}$ is bounded above by 6.

Limit: By __________________________, we know $\lim_{n\to\infty} a_n$ exists. Let $\lim_{n\to\infty} a_n = L$. Since

11.2 Series (page 707)

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前一節討論的無窮數列理論的目的是爲了研究無窮級數，因爲數字有無限多個，我們永遠無法確實地把所有數字都加起來，所以利用部份和還有極限的概念去理解無窮級數。

Definition 1 (page 707–708)**.** Let $\{a_n\}_{n=1}^{\infty}$ be an infinite sequence.

(1) The *partial sums* (部份和) of the sequence $\{a_n\}_{n=1}^{\infty}$ is defined as

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

These partial sums form a new sequence $\{s_n\}_{n=1}^{\infty}$ (部份和數列).

(2) An *infinite series* (or just a *series* 無窮級數) is denoted by

$$\sum_{n=1}^{\infty} a_n \overset{\text{def.}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n),$$

which means the limit of the partial sums of the sequence $\{a_n\}_{n=1}^{\infty}$.

(3) If the limit $\lim\limits_{n\to\infty} s_n = s$ exists (or convergent) as a finite number, then we say the series $\sum\limits_{n=1}^{\infty} a_n$ *convergent* (收斂), and the number s is called the *sum* of the infinite series $\sum\limits_{n=1}^{\infty} a_n$ (級數和).

(4) If the sequence $\{s_n\}_{n=1}^{\infty}$ is divergent, then the series $\sum\limits_{n=1}^{\infty} a_n$ is called *divergent* (發散).

☐ 微積分課程中感興趣的是「無窮級數」, 透過「部份和數列的極限」來定義無窮級數收斂或發散。

Example 2 (page 708)**.** In this chapter, we are *not* interested in the infinite *arithmetic series* (等差級數、算數級數):

$$\sum_{n=1}^{\infty} (a + (n-1)d) \overset{\text{def.}}{=} a + (a+d) + (a+2d) + \cdots + (a + (n-1)d) + \cdots ,$$

where each term is obtained from the preceding one by adding it by the *common difference* (公差) d. This is because the arithmetic series is convergent if and only if $a = 0$ and $d = 0$.

☐ 無窮等差級數除了每一項都是零的級數和收斂外, 其他情況都發散, 故不值得研究它。

Example 3 (page 709)**.** The *geometric series* (等比級數、幾何級數) is an infinite series

$$\sum_{n=1}^{\infty} ar^{n-1} \overset{\text{def.}}{=} a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots , \quad a \neq 0.$$

Each term is obtained from the preceding one by multiplying it by the *common ratio* (公比) r. We will discuss the convergence or divergence of the geometric series in the following theorem.

☐ 等比級數在無窮級數理論中佔了非常重要的角色, 務必徹底了解。

Theorem 4 (page 710)**.** *The geometric series*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots, \quad a \neq 0.$$

is convergent if $|r| < 1$ *and its sum is*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{if} \quad |r| < 1.$$

If $|r| \geq 1$*, the geometric series is divergent.*

oWYmQx95UIQ

等比級數是級數理論的標準模型，它的推論還有結果都很重要。 特別注意：無窮等比數列的收斂發散結果與無窮等比級數的收斂發散結果是不一樣的，要區分清楚。

Proof.

□

Example 5. Write the number $0.\overline{142857} = 0.142857142857\ldots$ as a ratio of integers (fraction).

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Solution.

用這個例子想清楚 $0.\overline{9}$ 和 1 兩者是否一樣？

Theorem 6 (page 713)**.** *If the series* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

若級數收斂，則原數列會趨近於零。這個定理只是級數收斂的必要條件，不是充要條件。我們比較常使用的是這個定理的否逆命題，也就是下面所寫的級數發散判別法。

Proof.

□

Test for Divergence (page 713)**.** *If* $\lim_{n\to\infty} a_n$ *does not exist or if* $\lim_{n\to\infty} a_n \neq 0$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

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Example 7 (page 713)**.** The *harmonic series* (調和級數) is an infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n} \overset{\text{def.}}{=} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots .$$

調和級數的發散證明是利用 11.1 的定理 7: 證明調和級數的某個部份和子數列發散, 則原部份和數列發散這個定理。

Show that it is divergent.

Proof.

□

□ 若 $\lim_{n\to\infty} a_n = 0$, 則級數 $\sum_{n=1}^{\infty} a_n$ 收斂與否仍舊無法判定。

例如: 比較調和級數 $\sum_{n=1}^{\infty} \frac{1}{n}$、等比級數 $\sum_{n=1}^{\infty} ar^{n-1}$ 或歐拉級數 $\sum_{n=1}^{\infty} \frac{1}{n^2}$。

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Theorem 8 (page 714)**.** *If* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are convergent series, then so are the series* $\sum_{n=1}^{\infty} c\, a_n$ *(where c is a constant),* $\sum_{n=1}^{\infty} (a_n + b_n)$*, and* $\sum_{n=1}^{\infty} (a_n - b_n)$*, and*

若用線性代數的語言來看這個定理, 則收斂的級數具有「線性」的性質。雖然級數的概念是源自於部份和數列的極限, 但是級數的乘與除並沒有相關的定理。 此外, 只有兩級數都收斂的時候才有定理的結論, 若有一個級數發散則結論不一定。

(a) $\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n$.

(b) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

(c) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

□ 各別的級數和 $\sum_{n=1}^{\infty} a_n$ 與 $\sum_{n=1}^{\infty} b_n$ 之「收斂」很重要。

□ 各項相加後得到的新的級數和與各別的級數和再相加相同。

□ 注意! $\sum_{n=1}^{\infty} a_n b_n \neq \sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n$。兩數列相乘的級數和不會等於各別級數和再相乘!

□ 級數和的收斂與否和前面有限項無關。

□ 若 $\sum_{n=1}^{\infty} a_n$ 收斂而 $\sum_{n=1}^{\infty} b_n$ 發散, 則 $\sum_{n=1}^{\infty} (a_n + b_n)$ 發散。(習題 11.2, #83。)

□ 若 $\sum_{n=1}^{\infty} a_n$ 與 $\sum_{n=1}^{\infty} b_n$ 發散, 則 $\sum_{n=1}^{\infty} (a_n + b_n)$ 不一定收斂也不一定發散。(習題 11.2, #84。)

11.3 The Integral Test and Estimates of Sums (page 719)

The Integral Test (page 721). *Suppose $f(x)$ is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,\mathrm{d}x$ is convergent. In other words,*

(a) *If $\int_1^{\infty} f(x)\,\mathrm{d}x$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.*

(b) *If $\int_1^{\infty} f(x)\,\mathrm{d}x$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.*

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瑕積分的收斂發散性可以幫助我們了解級數的收斂與發散。這個定理是等價敘述，瑕積分與級數同享收斂或發散的性質。積分判別法只適用於「正項級數」。

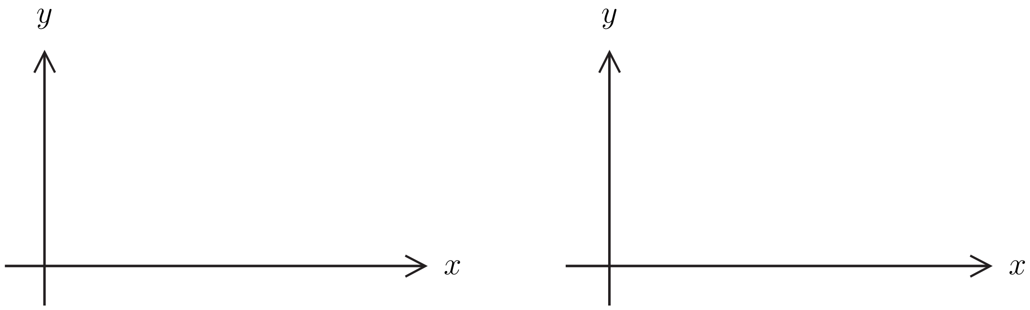

Figure 1: The integral test.

☐ 函數 $f(x)$ 必須「恆正」與「遞減」, 函數的連續性是要讓積分比較好處理。

☐ 定理使用時不見得要「從頭 $n = 1, x = 1$ 開始」; 收斂和發散和前面有限項無關。

☐ 定理只是說明瑕積分與級數享有相同的斂散性, 並不代表兩者具有相同的值。

Theorem 1 (page 721). *The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (p-級數) is convergent if $p > 1$ and divergent if $p \leq 1$.*

aRHTPX5D7XI

p-級數也是級數理論的標準模型，其論述與結論都必須確實理解。

Proof. <u>If $p < 0$</u>,

<u>If $p = 0$</u>,

<u>If $p > 0$</u>, consider $f(x) = \frac{1}{x^p}$, which is continuous, positive and decreasing on $[1, \infty)$. Since

☐

9jI7TEanSjg

Example 2 (page 722)**.** Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Solution.

例題示範用積分判別法驗證級數的收斂或發散。另一方面, 各位應該要體會到這個級數是發散的, 因爲分母對應到 $p = 1$, 而分子終將大於 1, 所以級數比 $\sum_{n=1}^{\infty} \frac{1}{n}$ 還要差。

☐ 先觀察當指標改成 x 時有沒有辦法用瑕積分驗證斂散性, 可以的話再逐一檢查條件。

Estimating the Sum of a Series, page 723

ERoRqkUnhV8

Suppose a series $\sum_{n=1}^{\infty} a_n$ is convergent by the Integral Test. We can also estimate the size of the *remainder* (餘項)

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots = \sum_{k=n+1}^{\infty} a_k.$$

利用積分判別法確定的收斂級數, 其級數和可以進行估計: 給定一個誤差之下, 可以確定要加的項數使得級數和與部份和之差小於給定的誤差。

Remainder Estimate for the Integral Test (page 718)**.** *Suppose $f(k) = a_k$, where $f(x)$ is a continuous, positive, decreasing function for $x \geq n$ and $\sum_{n=1}^{\infty} a_n$ is convergent. If $R_n = s - s_n$, then*

$$\int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq R_n \leq \int_{n}^{\infty} f(x)\,\mathrm{d}x. \tag{1}$$

If we add s_n to each side of the inequalities (1)*, because $s_n + R_n = s$, we get*

$$s_n + \int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq s \leq s_n + \int_{n}^{\infty} f(x)\,\mathrm{d}x.$$

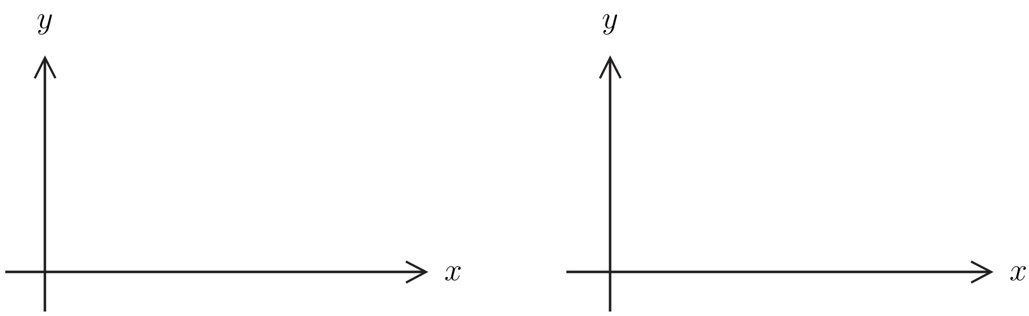

Figure 2: Remainder estimate for the Integral Test.

Example 3 (page 723). Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$. How many terms are required to ensure that the sum is accurate to within 0.005?

tXW4uKcot00

例題示範如何用積分判別法進行級數餘項估計。

Solution.

11.4 The Comparison Tests (page 727)

X0P1V6VewNE

比較判別法可以用來討論複雜級數之斂散性。但是比較定理只有單邊的性質，也就是說，大的級數收斂可推得小的級數收斂；小的級數發散可推得大的級數發散。而且， 它只適用於「正項級數」。

The Comparison Test (page 727)**.** *Suppose that* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are series with positive terms and* $a_n \leq b_n$ *for all* n*.*

(a) *If* $\sum_{n=1}^{\infty} b_n$ *is convergent, then* $\sum_{n=1}^{\infty} a_n$ *is also convergent.*

(b) *If* $\sum_{n=1}^{\infty} a_n$ *is divergent, then* $\sum_{n=1}^{\infty} b_n$ *is also divergent.*

Proof. Let $s_n = \sum_{k=1}^{n} a_k, t_n = \sum_{k=1}^{n} b_k$, and $t = \sum_{k=1}^{\infty} b_k$.

(a) Monotone: Since both series have positive terms, the sequences $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are increasing.

Bounded: Since $a_k \leq b_k$ for all k, we have $s_n \leq t_n \leq t$.

By the ________________________________, $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\sum_{n=1}^{\infty} a_n$ is divergent, then $s_n \to \infty$, thus $t_n \to \infty$. Therefore $\sum_{n=1}^{\infty} b_n$ diverges.

□

N7NT0rbbPzg

我們經常利用等比級數與 p-級數這兩個標準模型來判定其它級數的斂散性，所以標準模型的斂散性結果必須熟知。

Most of time we use p-series and geometric series for the purpose of comparison.

(1) p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$. It is convergent if ______ and divergent if ______.

(2) geometric series: $\sum_{n=1}^{\infty} ar^{n-1}$. It is convergent if ______ and divergent if ______.

Example 1. Show that the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is convergent.

Solution.

The Limit Comparison Test (page 729). *Suppose that* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are series with positive terms. If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c,$$

where c is a finite number and $c > 0$, *then either both series converge or both diverge.*

95E2eNPi1qw

極限比較判別法相較於比較判別法來說適用的層面更廣，因爲兩級數之間不需建立明確的大小關係，只要確定兩數列之比的極限爲正數，則兩級數享有一樣的斂散性。

Proof. Let m and M be positive numbers such that $m < c < M$. Since $\frac{a_n}{b_n}$ is close to c for large n, there is an integer N such that

$$m < \frac{a_n}{b_n} < M \Rightarrow mb_n < a_n < Mb_n \qquad \text{when } n > N.$$

By the ____________________, we know both series converge or both diverge. □

□ 比較判別法與極限比較判別法只適用於「正項級數」。

Example 2 (page 730). Determine whether the following series converges or diverges.

$$\text{(a)} \sum_{n=1}^{\infty} \frac{9^n}{3+10^n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}.$$

3fRmnVgSWBE

例題示範如何用極限比較判別法證明級數的斂散性。如何選取要比較的級數呢？想法是把一般項「最重要」的部份抓出來，這個概念與「等級」(order) 有關，會在之後的單元闡明。

Solution.

Exercise (page 726). Determine whether the following series converges or diverges.

$$\text{(a)} \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}} \qquad \text{(c)} \sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n}.$$

Estimating Sums, page 730

OO_YHbDvZg8

若用比較判別法得知級數收斂，則可進行誤差估計，誤差的精準度會依賴於比較的級數。

If we have used the Comparison Test to show that a series $\sum_{n=1}^{\infty} a_n$ converges by comparison with a series $\sum_{n=1}^{\infty} b_n$, then we may be able to estimate the sum $\sum_{n=1}^{\infty} a_n$ by comparing remainders.

Consider the remainder $R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$ and $T_n = t - t_n = b_{n+1} + b_{n+2} + \cdots$. Since $a_n \le b_n$ for all n, we have $R_n \le T_n$.

(1) If $\sum_{n=1}^{\infty} b_n$ is a p-series, we can estimate its remainder T_n as in Section 11.3.

(2) If $\sum_{n=1}^{\infty} b_n$ is a geometric series, we can sum it exactly.

LNUObzAzm5c

以下兩個例題分別示範比較判別法的餘項估計，其中一個是用 p-級數進行比較，另一個是用等比級數進行比較。

Example 3 (page 730)**.** Use the sum of the first 100 terms to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$. Estimate the error involved in this approximation.

Solution.

Example 4 (page 731)**.** Use $\sum_{n=1}^{10} \frac{\cos^2 n}{5^n} \doteq 0.07393$ to estimate the error of the sum of the series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{5^n}$.

Solution.

Exercise. Use $\sum_{n=1}^{10} \frac{1}{3^n+4^n} \doteq 0.19788$ to estimate the error of the sum of the series $\sum_{n=1}^{\infty} \frac{1}{3^n+4^n}$.

11.5 Alternating Series (page 732)

Definition 1 (page 732)**.** An *alternating series* (交錯級數) is a series whose terms are alternately positive and negative.

gIEY2_ZYvQQ

當級數的每一項正負交錯時， 其斂散性可以用交錯級數判別法測試。當忘掉正負符號的一般項遞減且趨近於零，則交錯級數收斂。

Example 2 (page 732)**.** Two examples of alternating series are

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$
$$\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1} = -\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots \qquad .$$

Alternating Series Test (page 727)**.** *If the alternating series*

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots, \quad \textit{where } b_n > 0,$$

satisfies

(a) $b_{n+1} \le b_n$ *for all* n

(b) $\lim_{n\to\infty} b_n = 0$,

then the series is convergent.

Figure 1: Alternating series test.

☐ 交錯級數只要忘掉符號的「某一項之後遞減」並且「趨近於零」, 則級數收斂。

Example 3 (page 734)**.** Determine whether the following series converges or diverges.

iPDMJLw-pGQ

注意到交錯級數判別法只是必要條件，當判別的條件不滿足時，定理不適用；必須要用別的理論判定交錯級數發散。

$$\text{(a)}\ \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} \qquad \text{(b)}\ \sum_{n=1}^{\infty}\frac{(-1)^{n}3n}{4n-1}.$$

Solution.

k0D1mz4TGro

這個例題要從中學到的是如何確定交錯級數忘掉符號的一般項爲遞減。將上學期的微分理論結合。

Example 4 (page 734). Test the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$ for convergence or divergence.

Solution.

Exercise. Test the series $\sum_{n=1}^{\infty}(-1)^n\left(e^{\frac{1}{n}}-1\right)$ for convergence or divergence.

Estimating Sums, page 735

sP9rRIVfLiI

交錯級數的誤差估計結果還蠻簡潔明瞭的，誤差不會超過第一個餘項。

Alternating Series Estimation Theorem (page 735). *If* $s=\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ *is the sum of an alternating series that satisfies*

(a) $b_{n+1}\le b_n$

(b) $\lim_{n\to\infty} b_n=0$,

then $|R_n|=|s-s_n|\le b_{n+1}$.

☐「好的」交錯級數 (滿足 (a) 與 (b)), 則級數和與有限項和之誤差只要看第一個餘項。

☐ 此定理只適用於「交錯級數」, 其他類型的級數不適用。

Example 5 (page 735). Find the sum of the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$ correct to three decimal place.

Solution. Since $\frac{1}{(n+1)!}=\frac{1}{(n+1)n!}<\frac{1}{n!}$ and $0\le\lim_{n\to\infty}\frac{1}{n!}\le\lim_{n\to\infty}\frac{1}{n}=0$, the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$ converges by the ____________________. By the Alternating Series Estimation Theorem we hope $|s-s_n|\le b_{n+1}<0.0005$, so $(n+1)!>2000$ and $n\ge 6$. Hence $s\approx s_6=\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}=0.368056\ldots\doteq 0.368$ correct to three decimal places with maximum error less than 0.001.

Exercise (page 736). How many terms of the series do we need to add in order to find the sum of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^6}$ correct to four decimal place?

11.6 Absolute Convergence and the Ratio and Root Tests (page 737)

Definition 1 (page 737-738)**.**

(1) A series $\sum_{n=1}^{\infty} a_n$ is called *absolutely convergent* (絕對收斂) if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

(2) A series $\sum_{n=1}^{\infty} a_n$ is called *conditionally convergent* (條件收斂) if it is convergent but not absolutely convergent.

Txpji7oaJLA

級數絕對收斂顧名思義是把每一項加絕對值之後的級數收斂，將每一項先加絕對值，它就形成正項級數。

Example 2 (page 737)**.**

(a) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent.

(b) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent.

Example 3. Determine the series $\sum_{n=1}^{\infty} (-1)^n \sin \frac{1}{n}$ is absolutely convergent, conditionally convergent, or divergent.

HBAZYlvpB2Q

上學期曾經學過的 $\lim_{x\to 0} \frac{\sin x}{x} = 1$ 應該要和這個例子聯想，得知 $\sin \frac{1}{n}$ 當 n 很大的時候和 $\frac{1}{n}$ 差不多。由此可預測級數的斂散性。

Solution.

Exercise. Determine the series (a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$ and (b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)}$ is absolutely convergent, conditionally convergent, or divergent.

Theorem 4 (page 738)**.** *If a series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent, then it is convergent.*

x_wgYkVW7Os

若級數絕對收斂，則原級數收斂，反之不一定成立。

Proof.

□

比值判別法的想法是類於於等比級數的操作, 後項比前項就像是公比的概念, 加絕對值再取極限之後會以 1 爲分界。注意到極限值若爲 1, 則判別法失效。

The Ratio Test (page 739)**.**

(a) *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent* (*and therefore convergent*).

(b) *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ *or* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

(c) *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, *the Ratio Test is inconclusive* (無法確定的)*; that is, no conclusion can be drawn about the convergence or divergence of* $\sum_{n=1}^{\infty} a_n$.

☐ 加上絕對值後, 級數的「行爲」被公比爲 r 的等比級數控制, 其中 $L < r < 1$。

5_EGI1__1Bk

Example 5 (page 740)**.** Determine whether the series is absolutely convergent, conditionally convergent, or divergent. (a) $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$ (b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2 2^n}{n!}$.

例題示範如何用比值判別法得知級數是絕對收斂或發散。

Solution.

☐ 帶有「指數」或「階乘」的級數, 比值法 (Ratio Test) 通常很好用。

☐ 帶有「多項式」、「有理函數」或帶有「三角函數」, 通常用比較判別法。

☐ 比值判別法無法確定的例子: $\sum_{n=1}^{\infty}\frac{1}{n}$ 發散, 而 $\sum_{n=1}^{\infty}\frac{1}{n^2}$ 收斂。

Exercise (page 743)**.** Determine whether the series is absolutely convergent, conditionally convergent, or divergent. (a) $\sum_{n=1}^{\infty}\frac{2^{n^2}}{n!}$ (b) $\sum_{n=1}^{\infty}\frac{(n!)^2}{(5n)!}$.

0fRNffD1IGw

The Root Test (page 741)**.**

根式判別法的想法也是來自於等比級數, 因爲將等比級數的一般項取絕對值開 n 次根號後就會出現公比。

(a) *If* $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent* (*and therefore convergent*).

(b) *If* $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$ *or* $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

(c) *If* $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$, *the Root Test is inconclusive* (無法確定的).

Example 6 (page 741)**.** Test the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$.

例題示範用根式判別法確定級數的斂散性。

Solution.

☐ 通常級數型如 $\sum_{n=1}^{\infty}(a_n)^n$ 可考慮用根式法 (Root Test)。

☐ 比值法比根式法重要一些 (11.8 之後)。

Exercise (page 743)**.** Determine whether the series is absolutely convergent, conditionally convergent, or divergent. (a) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$ (b) $\sum_{n=1}^{\infty}(\sqrt{n}-\sqrt{n-1})^{2n}$.

Rearrangements, page 742

If we rearrange the order of the terms in a finite sum, then the value of the sum remains unchanged. But it is *not* always the case for an infinite series.

9Skjoj7F2Hk

By a *rearrangement* of an infinite series $\sum_{n=1}^{\infty} a_n$ (更序級數) we mean a series obtained by simply changing the order of the terms. Formally, we will write $\sum_{\sigma(n)} a_{\sigma(n)}$ where $\sigma(n)$ is an one-to-one map from the natural number $\mathbb{N}$ to itself. For instance, a rearrangement of $\sum_{\sigma(n)} a_{\sigma(n)}$ could start as follows:

有限個數字相加則加法具有交換律。但是無限多個數字相加是否有交換律呢？這是非常深刻的數學問題，直到黎曼 (Riemann) 才確立了以下定理：絕對收斂的級數具有加法交換律；條件收斂的級數可以經過調整加到任何想加的數字。

$$a_2 + a_7 + a_3 + a_{32} + a_{15} + a_{10} + a_{200} + \cdots.$$

It turns out that

Theorem 7 (page 742)**.**

(a) *If* $\sum_{n=1}^{\infty} a_n$ *is an absolutely convergent series with sum* s*, then any rearrangement of* $\sum_{n=1}^{\infty} a_n$ *has the same sum* s*.*

(b) *If* $\sum_{n=1}^{\infty} a_n$ *is a conditionally convergent series and* r *is any real number whatsoever, then there is a rearrangement of* $\sum_{n=1}^{\infty} a_n$ *that has a sum equal to* r*.*

hAVP219aPs4

Example 8 (page 742)**.** Consider the alternating harmonic series

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots \tag{2}$$

例題示範調和級數(條件收斂)，記和爲 S，則經過順序的調整後，和變成 $\frac{3}{2}S$。

If we multiply this series by $\frac{1}{2}$ and insert 0 between the terms of new series, we get

$$\frac{1}{2}S = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \cdots \tag{3}$$

Now we add the series in (2) and (3) to get

$$\frac{3}{2}S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots \tag{4}$$

Notice that the series in (4) contains the same terms as in (2).

兩個絕對收斂的級數相乘，也有乘法對加法的分配律。

Theorem 9. *If $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} b_n$ are two absolutely convergent series with sum A and B, respectively, then the product series $\sum\limits_{n=0}^{\infty}\sum\limits_{k=0}^{n} a_k b_{n-k}$ and any rearrangement of $\sum\limits_{n=0}^{\infty}\sum\limits_{k=0}^{n} a_k b_{n-k}$ has a sum equal to AB.*

Appendix

aWmWSnWmzro

比值判別法的證明，設法和等比級數建立比較關係。

Proof of Ratio Test, page 739.

(a) Since $L < 1$, we can choose a number r such that $L < r < 1$. Since $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ and $L < r$, the ratio $\left|\frac{a_{n+1}}{a_n}\right|$ will eventually be less than r; that is, there exists an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| < r \quad\Longleftrightarrow\quad |a_{n+1}| < |a_n| r \quad \text{whwnever} \quad n \geq N.$$

In general, we get

$$|a_{N+k}| < |a_{N+k-1}| r < |a_{N+k-2}| r^2 < \cdots < |a_N| r^k \quad \text{for all} \quad k \geq 1.$$

By the Comparison Test, we know

$$\sum_{n=N+1}^{\infty} |a_n| = \sum_{k=1}^{\infty} |a_{N+k}| < \sum_{k=1}^{\infty} |a_N| r^k = \frac{|a_N| r}{1-r}.$$

Hence $\sum\limits_{n=1}^{\infty} |a_n|$ is convergent, and $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the ratio $\left|\frac{a_{n+1}}{a_n}\right|$ will eventually be greater than 1; that is, there exists an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \quad\Leftrightarrow\quad |a_{n+1}| > |a_n| \quad \text{whenever} \quad n \geq N.$$

Since $\lim\limits_{n\to\infty} a_n \neq 0$, the series $\sum\limits_{n=1}^{\infty} a_n$ diverges by the Test for Divergence.

$\square$

11.7 Strategy for Testing Series (page 739)

本文將討論如何利用直覺的方式判斷級數是絕對收斂 (Absolutely Convergent)、條件收斂 (Conditionally Convergent) 或發散 (Divergent), 以及歸納出幾個心得以快速找到證明級數收斂或發散的判別法。

-Z4Z5J7E3ZA

判斷級數的收斂或發散並沒有完整的標準程序 (Standard Operation Procedure), 以下只是提供幾個經驗分享。以下的原則大體上可以涵蓋各位將面臨到的 90% 的級數。剩下的 10% 算是比較特殊的級數, 例如第 25, 35, 36, 38, 50, 62, 65 題, 各位需額外花時間仔細研究其性質, 再將結果納入心得。

前面幾個單元都是介紹各種級數判別法該如何使用, 並寫出完整地數學論述。 但是隨便一個級數, 你要怎麼預判它是絕對收斂、條件收斂、還是發散? 並且要用哪個級數比較還有要用什麼定理論述? 我們可以透過等級 (order) 來了解級數收斂或發散的理論。各位應好好體會級數理論的奧義, 這是高等數學的一個很重要的概念。

(1) 心中一定要非常清楚以下兩類基本的級數收斂與發散:

- p-級數 (p-series) $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$: 此級數當 $p > 1$ 時收斂, 當 $0 < p \leq 1$ 時發散。
- 另一個是等比級數 (geometric series) $\sum\limits_{n=1}^{\infty} ar^{n-1}$: 此級數當 $|r| < 1$ 時收斂, 當 $|r| \geq 1$ 時發散。

(2) 利用等級 (order) 的觀念 "猜測" 級數是絕對收斂 (A.C.)、條件收斂 (C.C.) 或是發散 (Div.)。常見也常用的等級順序如下:

$$1 \ll \ln n \ll n^k \ll a^n \ll n! \ll n^n, \quad \text{其中 } k > 0, a > 1。$$

(3) 尋找適當的定理 (判別法), 通常來說,

- 只有單一類型, 或是不同類型的「相加」$\Rightarrow$ 比較判別法 (CT, LCT)。
- 兩種以上類型「相乘」, 或是帶有階乘 $\Rightarrow$ 比值法 (Ratio T)。
- 級數型如 $(b_n)^n \Rightarrow$ 根式法 (Root T)。
- 級數正負交錯 $\Rightarrow$ 交錯級數法 (AST)。
- 特殊函數, 例如 $\ln n$, 觀察它是否連續化之後可以積分 $\Rightarrow$ 積分法 (IT)。
- 發散 $\Rightarrow$ (DT), 除了 AST 以外的判別法都有可能用到。

(3) 剩下的 10% 會遇到比較特殊或不顯而易見的等級 (order), 必須重新理解, 並設法納入"心得"。

(4) 注意到 $\sin n, \cos n, \sin \frac{1}{n}, \cos \frac{1}{n}, \tan \frac{1}{n}$ 對待的方式不同, 可見第 14, 21, 22, 23, 24, 34, 45, 55, 73, 87 題的分析。

(5) 熟悉以下極限也有助於判斷級數收斂或發散:

$$\lim_{x\to 0} \cos x = 1 \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{\tan x}{x} = 1$$
$$\lim_{x\to \infty} \sqrt[x]{a} = 1 \qquad \lim_{x\to \infty} \sqrt[x]{x} = 1$$
$$\lim_{x\to \infty} \left(1 + \frac{1}{x}\right)^x = \mathrm{e} \qquad \lim_{x\to 0} (1 + x)^{\frac{1}{x}} = \mathrm{e}$$

11.7 Exercises and 11 Review

BGxlrFgH3ZU

試著利用等級的概念先預判級數是絕對收斂、條件收斂或是發散。然後再挑選適當的定理或判別法給予嚴格的論述。

Determine whether the series is conditionally convergent, absolutely convergent, or divergent. (page 743, 746)

1. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+3^n}$

2. $\displaystyle\sum_{n=1}^{\infty}\frac{(2n+1)^n}{n^{2n}}$

3. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+2}$

4. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+2}$

5. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2 2^{n-1}}{(-5)^n}$

6. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n+1}$

7. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$

8. $\displaystyle\sum_{k=1}^{\infty}\frac{2^k k!}{(k+2)!}$

9. $\displaystyle\sum_{k=1}^{\infty}k^2\mathrm{e}^{-k}$

10. $\displaystyle\sum_{n=1}^{\infty}n^2\mathrm{e}^{-n^3}$

11. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^3}+\frac{1}{3^n}\right)$

12. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k\sqrt{k^2+1}}$

13. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n n^2}{n!}$

14. $\displaystyle\sum_{n=1}^{\infty}\frac{\sin 2n}{1+2^n}$

15. $\displaystyle\sum_{k=1}^{\infty}\frac{2^{k-1}3^{k+1}}{k^k}$

16. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+1}{n^3+1}$

17. $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{2\cdot 5\cdot\cdots\cdot(3n+2)}$

18. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}-1}$

19. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{\sqrt{n}}$

20. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n^2-1}}{n^3+2n^2+5}$

21. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\cos\left(\frac{1}{n^2}\right)$

22. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{2+\sin k}$

23. $\displaystyle\sum_{n=1}^{\infty}\tan\left(\frac{1}{n}\right)$

24. $\displaystyle\sum_{n=1}^{\infty}n\sin\left(\frac{1}{n}\right)$

25. $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{\mathrm{e}^{n^2}}$

26. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+1}{5^n}$

27. $\displaystyle\sum_{k=1}^{\infty}\frac{k\ln k}{(k+1)^3}$

28. $\displaystyle\sum_{n=1}^{\infty}\frac{\mathrm{e}^{\frac{1}{n}}}{n^2}$

29. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\cosh n}$

30. $\displaystyle\sum_{j=1}^{\infty}(-1)^j\frac{\sqrt{j}}{j+5}$

31. $\displaystyle\sum_{k=1}^{\infty}\frac{5^k}{3^k+4^k}$

32. $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^n}{n^{4n}}$

33. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$

34. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+n\cos^2 n}$

35. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^{1+\frac{1}{n}}}$

36. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(\ln n)^{\ln n}}$

37. $\displaystyle\sum_{n=1}^{\infty}\left(\sqrt[n]{2}-1\right)^n$

38. $\displaystyle\sum_{n=1}^{\infty}\left(\sqrt[n]{2}-1\right)$

39. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^3+1}$

40. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{\left(n+\frac{1}{n}\right)^n}$

41. $\displaystyle\sum_{n=1}^{\infty}\frac{n^3}{5^n}$

42. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$

43. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2\sqrt{\ln n}}$

44. $\displaystyle\sum_{n=2}^{\infty}\ln\left(\frac{n}{3n+1}\right)$

45. $\sum_{n=1}^{\infty} \frac{\cos 3n}{1+(1.2)^n}$ **46.** $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$ **47.** $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{5^n n!}$ **48.** $\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{n^2 9^n}$

49. $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{\sqrt{n}}{n+1}$ **50.** $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$ **51.** $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-\frac{1}{3}}$ **52.** $\sum_{n=1}^{\infty} (-1)^{n-1} n^{-3}$

53. $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)3^n}{2^{2n+1}}$ **54.** $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{\ln n}$ **55.** $\sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{3}\right)}{n!}$ **56.** $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

57. $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$ **58.** $\sum_{n=2}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$ **59.** $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$ **60.** $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

61. $\sum_{n=1}^{\infty} \frac{n^{100}100^n}{n!}$ **62.** $\sum_{n=1}^{\infty} \frac{n!}{2^{n^2}}$ **63.** $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$ **64.** $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}-1}{k(\sqrt{k}+1)}$

65. $\sum_{n=1}^{\infty} \sqrt{n+1}\left(1-\cos\frac{\pi}{n}\right)$ **66.** $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ **67.** $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$ **68.** $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{n^4}{4^n}$

69. $\sum_{n=0}^{\infty} (\tan^{-1} n)^n$ **70.** $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(\ln n)^n}$ **71.** $\sum_{n=3}^{\infty} \frac{1}{\ln(n!)}$ **72.** $\sum_{n=2}^{\infty} \left(\frac{n}{\ln n}\right)^n$

73. $\sum_{n=1}^{\infty} n\tan\frac{1}{2^n}$ **74.** $\sum_{n=1}^{\infty} \frac{1}{(\ln(n+1))^n}$ **75.** $\sum_{n=1}^{\infty} \left(\frac{1}{n}-\sin\frac{1}{n}\right)$ **76.** $\sum_{n=1}^{\infty} \frac{n^{n-1}}{(2n^2+n+1)^{\frac{n+2}{2}}}$

77. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n-\ln n}$ **78.** $\sum_{n=2}^{\infty} \frac{(-1)^n\sqrt{n}}{n-1}$ **79.** $\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)!}{n^{n+1}}$ **80.** $\sum_{n=1}^{\infty} (-1)^n \ln\left(\frac{n+1}{n}\right)$

81. $1-\frac{1\cdot 3}{3!}+\frac{1\cdot 3\cdot 5}{5!}-\frac{1\cdot 3\cdot 5\cdot 7}{7!}+\cdots+(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(2n-1)!}+\cdots$

82. $\frac{2}{5}+\frac{2\cdot 6}{5\cdot 8}+\frac{2\cdot 6\cdot 10}{5\cdot 8\cdot 11}+\frac{2\cdot 6\cdot 10\cdot 14}{5\cdot 8\cdot 11\cdot 14}+\cdots$ **83.** $\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{n!}$ **84.** $\sum_{n=1}^{\infty} (-1)^n \frac{2^n n!}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}$

85. $\frac{1}{\ln 2}-\frac{1}{\ln 3}+\frac{1}{\ln 4}-\frac{1}{\ln 5}+\cdots$ **86.** $1+\frac{1+2}{1+2^2}+\cdots+\frac{1+n}{1+n^2}+\cdots$ **87.** $\sin\frac{\pi}{2}+\sin\frac{\pi}{2^2}+\cdots+\sin\frac{\pi}{2^n}+\cdots$

11.8 Power Series (page 746)

2kojP3VuFN4

前幾節所學的是無窮級數，現在要把函數的概念結合，變成每代入一個數都要問一次級數的收斂或發散。那些使級數收斂的點與級數和的對應關係變成函數的概念。冪級數較爲特別，它的一般項是 $c_n x^n$ 的形式。

Definition 1 (page 746)**.** A *power series* (冪級數) is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots ,$$

where x is a variable and the c_n's are constants called the *coefficients* (係數) of the series.

A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$$

whose *domain* (定義域) is the set of all x for which the series converges.

☐ 「冪級數」可想成是「多項式」的推廣 — 多了極限的運算。

☐ 「冪級數」是一個函數 $f(x)$, 函數的定義域是級數收斂所成的集合。

Example 2 (page 746)**.** If $c_n \equiv 1$, the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots ,$$

which converges when ______________ and diverges when __________.

我們也可以討論中心移到 $x = a$ 的冪級數。注意這裡有一些記號上的約定。

Definition 3 (page 747)**.** A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a *power series in* $(x-a)$ (以 $(x-a)$ 形式的冪級數) or a *power series centered at* a (以 a 爲中心的冪級數) or *power series about* a (關於 a 的冪級數).

☐ 約定 $(x-a)^0 \equiv 1$, 即使 $x = a$ 也是如此。

☐ 任何關於 a 的冪級數, 必在 $x = a$ 收斂, 所以冪級數的定義域是非空集合。

S3GVhXpAEZo

冪級數的收斂定理，直接利用比值判別法的結果順勢而得。注意端點的收斂性總是要另外討論。

Theorem 4 (page 749)**.** *For a given power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *there are only three possibilities:*

(a) *The series converges only when* $x = a$.

(b) *The series converges for all* x.

(c) *There is a positive number* R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$. (注意此定理還不完整, 端點收斂行爲因級數而異。)

Definition 5 (page 749)**.**

(1) The number R in case (c) is called the *radius of convergence* (收斂半徑) of the power series.

(2) By convention, the radius of convergence is $R = 0$ in case (a) and $R = \infty$ in case (b).

(3) The *interval of convergence* (收斂區間) of a power series is the interval that consists of all values of x for which the series converges. When x is an *endpoint* (端點) of the interval, that is, $x = a \pm R$, anything can happen – the interval of convergence could be

$$(a - R, a + R) \qquad (a - R, a + R] \qquad [a - R, a + R) \qquad [a - R, a + R].$$

Example 6 (page 747)**.** Find the interval of the convergence of the following series:

(a) $\displaystyle\sum_{n=0}^{\infty} n! x^n$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ (Bessel function of order 0) (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n$.

uUmfwcHLv7s

這裡的學習，除了要會確實論述冪級數的收斂或發散，也要會從等級 (order) 的概念去感受冪級數的特性。

Solution.

11.9 Representations of Functions as Power Series (page 752)

YMXmwlzqXZY

In this section, we learn how to represent certain types of functions as sums of power series. We will see that it is useful for integrating functions that don't have elementary antiderivatives, for solving differential equations, and for approximating functions by polynomials.

這個單元要介紹幾個從等比級數出發透過一些基本運算下就可以順勢寫出的冪級數。 這裡必須先對「公式」確實理解才有辦法做出千變萬化的結果。

Example 1 (page 752). Recall that the geometric series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n \qquad \text{if} \quad |x| < 1.$$

We can express the following functions by manipulating geometric series:

(1) $\dfrac{1}{1+x^2} =$

(2) $\dfrac{x}{2+x} =$

Differentiation and Integration of Power Series, page 754

冪級數在收斂區間的內部有很好的操作特性，就像多項式一般地操作微分與積分。注意 (a) 的情況，微分後養成習慣把取和的下指標改成 $n = 1$, 在之後的一些計算上比較不會產生困擾。

Theorem 2 (page 754). *If the power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has radius of convergence* $R > 0$, *then the function* $f(x)$ *defined by*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ *and*

(a) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \displaystyle\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.$

(b) $\displaystyle\int f(x)\,\mathrm{d}x = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}.$

☐「冪級數」和「多項式」一樣, 可以逐項微分、逐項積分, 而且收斂「半徑」不變。
(term-by-term differentiation and integration)

☐ 重新看待定理中的 (a), (b), 對於收斂的冪級數:

PKUbCdvWTrg

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n\right) = \displaystyle\sum_{n=0}^{\infty} \dfrac{\mathrm{d}}{\mathrm{d}x}(c_n(x-a)^n)$ 「微分」和「求和、極限」可交換。

(b) $\displaystyle\int \left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) \mathrm{d}x = \sum_{n=0}^{\infty} \int c_n(x-a)^n \mathrm{d}x$ 「積分」和「求和、極限」可交換。

用這樣的表示法可以把定理看得很清楚, 它是微分或積分與求和之間的互換, 在冪級數的情況下是合法的。

☐「收斂半徑」相同不代表「收斂範圍」相同 (端點的收斂性會改變), 所以端點一律重新檢查。

Example 3 (page 745)**.** Express the following function as a power series and find its interval of convergence.

這三個例子非常經典, 特別是對數函數與反正切函數, 它們的微分正好可以和等比級數公式對應, 所以可以順勢地改寫。

$$(1)\ f(x) = \frac{1}{(1-x)^2} \qquad (2)\ g(x) = \ln(1+x) \qquad (3)\ h(x) = \tan^{-1} x.$$

Solution.

0njTFhZc8yc

有些級數的求和問題可以用冪級數的觀點處理，試著透過係數與次方中的 n 的關係推理出它和冪級數及其微分或積分的關聯。

Example 4 (page 758)**.** Find the sum of each of the following series.

$$(1)\ \sum_{n=1}^{\infty} nx^n, \quad |x| < 1 \qquad\qquad (2)\ \sum_{n=1}^{\infty} \frac{n}{2^n}.$$

Solution.

aDV_1SIgMKk

對於那些積不出來的函數或是積分處理很困難的函數，若要尋求定積分或瑕積分的值，退而求其次地，在允許一個誤差之下，改用冪級數展開，研究冪級數要加到多少項之下的積分值與真正值之間的差在誤差範圍內，用這種方式理解積分之意義。

Example 5 (page 750)**.** Evaluate $\int \frac{1}{1+x^7}\,\mathrm{d}x$ as a power series and approximate $\int_0^{0.5} \frac{1}{1+x^7}\,\mathrm{d}x$ correct to within 10^{-7}.

Solution. We express the integrand and then integrate term by term:

$$\frac{1}{1+x^7} =$$

$$\int \frac{1}{1+x^7}\,\mathrm{d}x =$$

This series converges for __________, that is ________.

$$\int_0^{0.5} \frac{1}{1+x^7}\,\mathrm{d}x =$$

When we choose $n = 3$, by the Alternating Series Estimation Theorem, the error is smaller than the term with $b_4 = \frac{1}{29\cdot 2^{29}} \approx 6.4 \times 10^{-11}$, so we have

$$\int_0^{0.5} \frac{1}{1+x^7}\,\mathrm{d}x \approx$$

11.10 Taylor and Maclaurin Series (page 759)

In this section, we will answer two questions: Which functions have power series representation? How can we find such representation?

02stBBo5Wvs

First, suppose that a smooth function $f(x)$ can be represented by a power series:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots, \qquad \text{if } |x-a| < R. \tag{1}$$

這一節的目的是要研究其它函數能不能重新表示成冪級數的形式。研究的方法第一步是「假設」函數可以順利地寫成冪級數，先得到冪級數每一項係數的表達式。這一節是級數理論的重頭戲，應徹底理解與體會。

- Put $x = a$, then we get ________.
- Since $f'(x) =$ ____,

  we put $x = a$ and get ________.
- Since $f''(x) =$ ________________, we put $x = a$ and get ________.
- By induction, since $f^{(k)}(x) =$ ________________, we have ________.

Theorem 1 (page 759)**.** *If $f(x)$ has a power series representation (expansion) at a:*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \text{ for } |x-a| < R$$

then its coefficients are given by the formula $c_n = \dfrac{f^{(n)}(a)}{n!}$.

Definition 2 (page 760)**.** Given a smooth function $f(x)$, define the *Taylor series of the function $f(x)$ at a* (or *about a* or *centered at a*) (函數 $f(x)$ 在 $x = a$ 處的泰勒級數) by

$$T(x) \overset{\text{def.}}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots. \tag{2}$$

由剛才的討論，對於函數在某一點可以無限次微分的函數，我們就可以定義泰勒級數或是馬克勞林級數。之後的篇幅是要研究原函數及其泰勒級數或馬克勞林級數的關係。

For the special cases $a = 0$ the series (2) becomes

$$M(x) \overset{\text{def.}}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots.$$

This case the function $M(x)$ is given the special name *Maclaurin series* (馬克勞林級數).

通常我們會先研究馬克勞林級數，對於中心不同的泰勒級數之情況，只要再知道一些平移的理論或轉換式，就可以把式子改寫。

☐ 由前面討論知道:「若 $f(x)$ 可表示成冪級數時」, 則 $f(x)$ 和它的泰勒級數 $T(x)$ 一致。

☐ 我們必須追問 (研究): 有哪些函數「可以」寫成冪級數?(存在函數無法表示成冪級數。)

4VY_jxCsk5g

這個例題是先求出指數函數的馬克勞林級數。注意到例題中的後半段討論收斂性只是在了解冪級數的定義域而已。

Example 3 (page 760)**.** Find the Maclaurin series of the function $f(x) = \mathrm{e}^x$ and its radius of convergence.

Solution. Since $f^{(n)}(x) =$ ____, we know that $f^{(n)}(0) =$ ________ for all $n \in \mathbb{N}$ or $n = 0$. Therefore the Maclaurin series of $f(x) = \mathrm{e}^x$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n =$$

To find the radius of convergence, we let $a_n =$ _____, then

$$\left| \frac{a_{n+1}}{a_n} \right| =$$

By the ____________, the radius of convergence is _______.

xvg6yDTPmq8

給了函數及其泰勒級數， 現在要開始研究兩者是否相等。首先把泰勒級數分成兩部份，一個是 n 階泰勒多項式，另一部份是餘項。

Question 4 (page 761)**.** Under what circumstances is a function equal to the sum of its Taylor series? In other words, if $f(x)$ has derivatives of all orders, when is it true that

$$f(x) \overset{?}{=} T(x) \overset{\text{def.}}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \overset{\text{def.}}{=} \lim_{n\to\infty} T_n(x),$$

where

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{3}$$

Definition 5 (page 761)**.**

(a) The polynomial $T_n(x)$ in (3) is called *n-th degree Taylor polynomial of $f(x)$ at a* ($f(x)$ 在 $x = a$ 的 n-階泰勒多項式).

(b) Define the *remainder* (餘項) of the Taylor series as $r_n(x) \overset{\text{def.}}{=} f(x) - T_n(x)$.

這個定理告知函數及其泰勒級數相等的等價條件是餘項趨近於零。 如果你把事情想清楚的話，就會覺得這個定理是蠻顯然的。

Theorem 6 (page 761)**.** *A smooth function $f(x) = T(x)$ on the interval $|x-a| < R$ if and only if $\lim_{n\to\infty} r_n(x) = 0$ for $|x-a| < R$.*

Proof. (⇒) Since $f(x) = \lim_{n\to\infty} T_n(x)$ and $r_n(x) = f(x) - T_n(x)$, we have

$$\lim_{n\to\infty} r_n(x) = \lim_{n\to\infty} (f(x) - T_n(x)) = f(x) - \lim_{n\to\infty} T_n(x) = f(x) - f(x) = 0.$$

(⇐) Conversely, since $\lim_{n\to\infty} r_n(x) = 0$ and $T_n(x) = f(x) - r_n(x)$, we have

$$T(x) = \lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} (f(x) - r_n(x)) = f(x) - \lim_{n\to\infty} r_n(x) = f(x) - 0 = f(x).$$

□

☐ 想清楚: 函數是否與其泰勒級數「相同」, 和泰勒級數的「收斂範圍」是兩回事。

☐ 定理得知: 函數與其泰勒級數在其收斂範圍內「相等」的等價條件是「餘項趨近於零」。

Question 7 (page 762)**.** How do we show that $\lim_{n\to\infty} r_n(x) = 0$ for a specific function $f(x)$?

Theorem 8. *Suppose that $f(x)$ has continuous derivative at $x = a$ up to $n+1$ order, then*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + r_n(x) = T_n(x) + r_n(x),$$

where $r_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, *$c$ is a number between a and x.*

903ROGCtJ1M

從前一個定理知道：函數及其泰勒級數相等的等價條件是餘項趨近於零。但是實際給了函數，要怎麼證明餘項趨近於零，這反而是相當困難的問題。這一頁主要是在講如何估計餘項。先將餘項（原本是無限多項相加）表示成一個簡單的型式；再得到下面的泰勒不等式。

Proof. Without loss of generality, we assume $a < x$. Consider the function

$$g(t) = f(x) - f(t) - \frac{f'(t)}{1!}(x-t) - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n,$$

then $g(t)$ is continuous on $[a, x]$, and

$$\begin{aligned} g'(t) &= -\sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k - \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}\,k(x-t)^{k-1}(-1) \\ &= -\sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} \\ &= -\sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{k=0}^{n-1} \frac{f^{(k+1)}(t)}{k!}(x-t)^k = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n. \end{aligned}$$

Let $h(t) = (x-t)^{n+1}$, by the Cauchy Theorem (generalized Mean Value Theorem), then there exists $c \in (a, x)$ such that

$$\frac{g'(c)}{h'(c)} = \frac{g(x)-g(a)}{h(x)-h(a)} \Rightarrow \frac{-\frac{f^{(n+1)}(c)(x-c)^n}{n!}}{-(n+1)(x-c)^n} = \frac{0 - r_n(x)}{0 - (x-a)^{n+1}},$$

so

$$r_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

□

□ 想成是「均值定理」的高階版本，餘項形式和泰勒多項式一樣，只是高次微分處代入 c。

Once we have this expression of the remainder, we can estimate it by the following theorem.

Taylor's Inequality (page 762)**.** *If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder $r_n(x)$ of the Taylor series satisfies the inequality*

$$|r_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \right| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \textit{for} \quad |x-a| \le d.$$

泰勒不等式的一個重點是函數 $n+1$ 次微分後的絕對值小於等於 M，這個 M 不能和 n 有關，這樣才有機會用階乘去控制指數而得到餘項趨近於零。

nrcD9xhLLjI

在認識函數及其泰勒級數相等的等價條件後，現在要開始研究基本函數及其泰勒級數的關係。先以指數函數為例，確實驗證兩者對所有實數都一致。此外，也用中心為 2 的例子了解指數函數在不同點的泰勒級數和原函數的關係。

Example 9 (page 763).

(1) Prove that e^x is equal to the sum of Maclaurin series.

(2) Find the Taylor series for $f(x) = e^x$ at $a = 2$.

Solution.

(1) If $f(x) = e^x$, then $f^{(n)}(x) = e^x$ for all $n \in \mathbb{N}$. Given $x \in \mathbb{R}$, there is a positive number d such that $|x| \le d$. Since $|f^{(n+1)}(x)| = e^x \le e^d$, we get

$$|r_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| \le \underline{\hspace{3cm}} \quad \text{for} \quad |x| \le d.$$

Notice that e^d is a number independent of n, so we have

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} =$$

By the Squeeze Theorem $\lim\limits_{n\to\infty} r_n(x) = 0$, and $e^x = \sum\limits_{n=0}^{\infty} \frac{1}{n!}x^n$ for all $x \in \mathbb{R}$.

(2) We have $f^{(n)}(2) = e^2$, so the Taylor series for $f(x) = e^x$ at $x = 2$ is

Another viewpoint is ________________________.

bNdXszACQ3s

這個例子是探討正弦函數及其馬克勞林級數的關係，結論也是非常好，處處收斂且相等。

Example 10 (page 764). Find the Maclaurin series for $f(x) = \sin x$. Prove that it represents $\sin x$ for all x.

Solution. We compute for $k = 0, 1, 2, 3, \ldots$,

$f^{(4k)}(x) =$ $\quad f^{(4k+1)}(x) =$ $\quad f^{(4k+2)}(x) =$ $\quad f^{(4k+3)}(x) =$

$f^{(4k)}(0) =$ $\quad f^{(4k+1)}(0) =$ $\quad f^{(4k+2)}(0) =$ $\quad f^{(4k+3)}(0) =$

so the Maclaurin series for $f(x) = \sin x$ is

Since $f^{(n+1)}(x)$ is $\pm\sin x$ or $\pm\cos x$, we know that $|f^{(n+1)}(x)| \le 1$ for all $x \in \mathbb{R}$. By Taylor's Inequality:

$$|r_n(x)| =$$

Since $\lim\limits_{n\to\infty}$ ________________________, we have $\lim\limits_{n\to\infty} r_n(x) = 0$ for all $x \in \mathbb{R}$ by the Squeeze Theorem. Thus $\sin x$ is equal to the sum of its Maclaurin series $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$.

Example 11 (page 764–765)**.**

(1) Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $x = \frac{\pi}{3}$.

(2) Find the Maclaurin series for $\cos x$.

(3) Find the Maclaurin series for $x \cos x$.

1VqXTH3fXfY

餘弦函數與其泰勒級數的關係，也可以仿照之前的方法再操作一次。而這個例題要示範的是透過一些三角函數的關係式還有冪級數的逐項微分逐項積分理論求得。後者的處理將有助於快速變化出更多函數的泰勒級數，免於總是土法煉鋼般地枯燥討論。

Solution. We have for $k = 0, 1, 2, 3, \ldots$

$f^{(4k)}(x) =$	$f^{(4k+1)}(x) =$	$f^{(4k+2)}(x) =$	$f^{(4k+3)}(x) =$
$f^{(4k)}(\frac{\pi}{3}) =$	$f^{(4k+1)}(\frac{\pi}{3}) =$	$f^{(4k+2)}(\frac{\pi}{3}) =$	$f^{(4k+3)}(\frac{\pi}{3}) =$

(1) The Taylor series at $\frac{\pi}{3}$ is

(2) Instead of computing derivatives and substituting in the Maclaurin series for $\cos x$, we can differentiate the Maclaurin series for $\sin x$:

$$\cos x =$$

Since the Maclaurin series for $\sin x$ converges for all x, the differential series for $\cos x$ also converges for all x.

(3) We can multiply the series for $\cos x$ by x:

$$x \cos x =$$

Example 12 (page 766)**.** Find the Maclaurin series for $f(x) = (1 + x)^m$, where m is *any real number.*

s7gCTjaAgkI

現在要討論的是二項式函數及其泰勒級數的關係。第一步仍然是要先把二項式函數的泰勒級數表示出來。注意到二項式函數的次方 m 可以是任何的實數。

Solution.

Therefore the Maclaurin series for $f(x) = (1 + x)^m$ is

接下來要確定由二項式函數對應到的馬克勞林級數之定義域，也就是級數的收斂性。收斂半徑較容易理解，但是端點的收斂情況與 m 有關，較爲複雜，將留到這一章的最後補充說明。

Example 13 (page 766)**.** Find the radius of convergence of the *binomial series* (二項式級數，從上一個例子推得) $\sum_{n=0}^{\infty} \frac{m(m-1)\cdots(m-n+1)}{n!}x^n$.

Solution. <u>If m is a nonnegative integer</u>, then the terms are eventually 0 and so the series is finite. <u>For other values of m</u>, if the n-th term is a_n, then

$$\left|\frac{a_{n+1}}{a_n}\right| =$$

By the ______________, the binomial series converges if ________ and diverges if ________, and the radius of convergence is ________.

將函數稱爲二項式函數的原因是其級數的係數將組合數 C_n^m 的概念推廣，這時 m 可以允許是任何實數。

The Binomial Series (page 767)**.** *If m is any real number and $|x| < 1$, then*

$$(1+x)^m = \sum_{n=0}^{\infty} C_n^m x^n = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots.$$

The interval of convergence depends on m: $(-1,1)$ if $m \le -1$; $(-1,1]$ if $-1 < m < 0$; $[-1,1]$ if $m > 0$.

□ 直接估計餘項趨近於零比較麻煩，有其他的方法證明二項式函數與二項式級數「相同」。

Definition 14 (page 766)**.** Numbers $C_n^m = \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}$ are called *binomial coefficients* (二項式係數). Remark that $C_0^m \equiv 1$ for all $m \in \mathbb{R}$.

UMOZvHfMms4

Example 15 (page 767)**.** Find the Maclaurin series for $g(x) = \frac{1}{\sqrt{4-x}}$ and its radius of convergence.

例題以 $m = -\frac{1}{2}$ 的二項式函數討論其馬克勞林級數。

Solution. We rewrite $f(x)$ in a form where we can use the binomial series:

$$\frac{1}{\sqrt{4-x}} =$$

Using the binomial series with $m =$ _____ and with x replaced by _____, we have

$$\frac{1}{\sqrt{4-x}} =$$

The series converges if ________________, so the radius of convergence is _____.

Important Maclaurin series and their radii of convergence

ndu0zmvTjXc

這個部份總結基本函數與其泰勒級數的關係。我們可以透過等級、奇偶性、在原點附近的行爲把這七個函數的泰勒級數熟記。

(1) $\displaystyle \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$

(2) $\displaystyle e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$

(3) $\displaystyle \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$

(4) $\displaystyle \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$

(5) $\displaystyle \tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$

(6) $\displaystyle \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$

(7) $\displaystyle (1+x)^m = \sum_{n=0}^{\infty} C_n^m x^n = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!} + \cdots \qquad R = 1$

Example 16 (page 768)**.** Find the sum of the series

$$\frac{1}{1\cdot 2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \cdots .$$

_XBE_z7S114

現在要開始實際應用，從泰勒及數的觀點重新理解微積分理論。級數和的問題也可以與函數的泰勒級數進行聯想。

Solution.

Example 17 (page 769)**.** Evaluate $\displaystyle \lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.

極限問題也可以重新認識，用泰勒級數的觀點了解函數的基本行爲，可以把極限看得更清楚。

Solution. Using the Maclaurin series for e^x, we have

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} =$$

$$=$$

because power series are continuous functions.

Multiplication and division of power series, page 770

4rSByKcMCfk

兩函數相乘或相除之泰勒級數也可以直接透過像多項式那樣直接四則運算求得。

Example 18 (page 770)**.** Find the first three nonzero terms in the Maclaurin series for (1) $e^x \sin x$ and (2) $\tan x$.

Solution.

M1VCJ93Oaac

例題中的積分是積不出來的，也就是不可能透過積分技巧把不定積分的結果用初等函數的方式表達。然而這個積分在統計、物理還有高等的數學理論有著非常重要的地位。對於這個積分的認識，退而求其次的方法就是將函數改用泰勒級數表示並進行誤差估計。

Example 19 (page 768)**.**

(1) Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

(2) Evaluate $\int_0^1 e^{-x^2}\,dx$ correct to within an error of 0.001.

Solution.

(1) We replace x with $-x^2$ in the series for e^x and get, for all $x \in \mathbb{R}$,

$$e^{-x^2} =$$

We integrate term by term: $\int e^{-x^2}\,dx =$ ______________________________ .

The series is convergent __________ because e^{-x^2} is convergent __________.

(2) We compute

$$\int_0^1 e^{-x^2}\,dx =$$

$$=$$

$$\approx$$

The Alternating Series Estimation Theorem shows that the error is less than

11.11 Applications of Taylor Polynomials (page 774)

In this section we explore some applications of Taylor polynomials. Computer scientists like them because polynomials are the simplest of functions. Physicists and engineers use them in such fields as relativity, optics, blackbody radiation, electric dipoles, the velocity of water waves, and building highways across a desert.

gaZEsOi5haI

這一單元將討論泰勒級數的各式應用，首先要把泰勒級數的概念與上學期所學的微分 (differential) 結合。微分只是一階或線性的近似，而泰勒多項式就是高階或是多項式的近似。

Approximating Functions by Polynomials, page 774

Recall that the *linear approximation* of $f(x)$ at $x = a$ (in section 3.10):

$$f(x) \approx f(a) + f'(a)(x-a) \tag{1}$$

Right hand side of (1), called the *linearization* of $f(x)$ at $x = a$, is the first-degree Taylor polynomial $T_1(x)$. If $f(x)$ is the sum of its Taylor series, then $T_n(x) \to f(x)$ as $n \to \infty$, and so $T_n(x)$, nth-degree Taylor polynomial of $f(x)$ at $x = a$, can be used as an approximation to $f(x)$:

$$f(x) \approx T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

When using a Taylor polynomial $T_n(x)$ to approximate a function $f(x)$, we have to ask that how good an approximation is it? How large should we take n to be in order to achieve a desired accuracy? To answer these questions we need to look at the absolute value of the remainder $|r_n(x)| = |R_n(x)| = |f(x) - T_n(x)|$. Here we remark that if $f(x)$ is the sum of its Taylor series, then $r_n(x) = R_n(x)$.

There are three possible methods for estimating the size of the error:

(1) If the series is an alternating series, we can use the Alternating Series Estimation Theorem.

(2) In all cases we can use Taylor Inequality: If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then

$$|r_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for} \quad |x-a| \le d.$$

(3) If a graphing device is available, we can use it to graph (estimate) $|R_n(x)|$.

Example 1. Desmos Graphing Calculator is a free, online, graphing calculator:

`https://www.desmos.com/calculator`
`https://desmos.s3.amazonaws.com/Desmos_User_Guide.pdf`

GuYLS6OySIU

我們也可以用數學繪圖軟體透過圖形的方法感受函數及其泰勒級數的相關性。看圖形變化時應著重於他們的近似範圍以及近似程度。

We will illustrate Taylor polynomial approximations by Desmos Calculator with some important examples.

Q8xluK-BGQE

在以前沒有計算機的時代，用泰勒級數估計一些特別的數字顯得非常重要，因爲它是一個用多項式還有極限的方法認識一些數學量。

Example 2 (page 775)**.**

(a) Approximate $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.

(b) How accurate is this approximation when $7 \le x \le 9$?

Solution.

(a) We compute

$$f(x) = \qquad f'(x) = \qquad f''(x) = \qquad f'''(x) =$$
$$f(8) = \qquad f'(8) = \qquad f''(8) =$$

So the desired approximation is

$$\sqrt[3]{x} \approx T_2(x) =$$
$$=$$

(b) We can use Taylor's Inequality with $n = 2$ at $a = 8$:

$$|r_2(x)| \le$$
$$\le$$

Thus, if $7 \le x \le 9$, the approximation in (a) is accurate to within ________.

VYKj43S7HpE

三角函數的估算也可以用泰勒級數的方法求得。現在雖然我們可以直接按計算機求得其精確值，但是計算機的構造原理是來自於泰勒級數的理論。

Example 3 (page 776)**.** What is the maximum error possible in using the approximation $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$ when $-0.3 \le x \le 0.3$? Use this approximation to find $\sin 12°$ correct to six decimal places.

Solution. Notice that the Maclaurin series $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ is alternating for all $x \ne 0$, and the successive terms decrease in size because $|x| < 1$, so we can use the ________________________________. The error in approximating $\sin x$ by the first three terms of its Maclaurin series is at most

$$\left|\frac{x^7}{7!}\right| = \frac{|x|^7}{5040} \le$$

To find $\sin 12°$, we first *convert to radian measure*:

$$\sin 12° = \sin\left(12 \cdot \frac{\pi}{180}\right) = \sin\left(\frac{\pi}{15}\right)$$
$$\approx$$

Thus, correct to six decimal places, $\sin 12° \approx$ ________.

Applications to Physics, page 778

Example 4 (page 778)**.** In Einstein's theory of special relativity the mass of an object moving with velocity v is

-cXS6MnYExU

愛因斯坦的狹義相對論與牛頓力學的關係也可以用泰勒級數的方法理解: 愛因斯坦理論在微觀尺度下與牛頓力學相當。

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}},$$

where m_0 is the mass of the object when at rest and c is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest: $K = mc^2 - m_0c^2$.

(a) Show that when v is very small compared with c, this expression for K agrees with classical Newtonian physics: $K = \frac{1}{2}m_0v^2$.

(b) Use Taylor's Inequality to estimate the difference in these expressions for K when $|v| \le 100\,\mathrm{m/s}$.

Solution.

(a) Using the expressions given for K and m, we get

$$K = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0c^2 = m_0c^2\left(\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} - 1\right).$$

With $x = -\frac{v^2}{c^2}$, the Maclaurin series for $(1+x)^{-\frac{1}{2}}$ is a binomial series with $m = -\frac{1}{2}$. Therefore we have

$$\begin{aligned}(1+x)^{-\frac{1}{2}} &= 1 - \frac{1}{2}x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3 + \cdots \\ &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots,\end{aligned}$$

and

$$K = m_0c^2\left(\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right) - 1\right) = m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right).$$

If v is much smaller than c, then all terms after the first are very small when compared with the first term. If we omit them, we get

$$K = m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2}\right) = \frac{1}{2}m_0v^2.$$

(b) Let $f(x) = m_0c^2\left((1+x)^{-\frac{1}{2}} - 1\right)$ with $x = -\frac{v^2}{c^2}$. We can use Taylor's Inequality to write

$$r_1(x) = \frac{f''(\tilde{c})}{2!}x^2, \quad \text{where } -\frac{v^2}{c^2} \le \tilde{c} \le 0.$$

Since $f''(x) = \frac{3}{4}m_0c^2(1+x)^{-\frac{5}{2}}$ and we are given that $|v| \le 100\,\mathrm{m/s}$, so

$$|f''(\tilde{c})| = \frac{3m_0c^2}{4\left(1+\tilde{c}\right)^{\frac{5}{2}}} \le \frac{3m_0c^2}{4\left(1-\frac{100^2}{c^2}\right)^{\frac{5}{2}}}.$$

Thus, with $c = 3\cdot 10^8\,\mathrm{m/s}$,

$$|r_1(x)| = \frac{1}{2}\cdot\frac{3m_0c^2}{4\left(1-\frac{100^2}{c^2}\right)^{\frac{5}{2}}}\cdot\frac{100^4}{c^4} < (4.17\cdot 10^{-10})m_0.$$

So when $|v| \le 100\,\mathrm{m/s}$, the magnitude of the error in using the Newtonian expression for kinetic energy is at most $(4.17\cdot 10^{-10})m_0$.

Appendix

tnkfRwtE4UI

這個例子在數學上的發展很重要，這個函數的建構告知其泰勒級數與原函數除了中心以外都不相等。由 (a) 到 (d) 的討論知道這個函數的馬克勞林級數是恆爲零的函數。既然如此，則不論 n 爲何，餘項和原函數一樣，所以餘項在非中心點的地方不可能趨近於零。

Example 5. Consider the function

$$f(x) = \begin{cases} \mathrm{e}^{-\frac{1}{x^2}} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

(a) The function $f(x)$ is continuous on $\mathbb{R}$ because

$$\lim_{x\to 0}\mathrm{e}^{-\frac{1}{x^2}} = \lim_{y\to\pm\infty}\mathrm{e}^{-y^2} = \lim_{y\to\pm\infty}\frac{1}{\mathrm{e}^{y^2}} = 0 = f(0),$$

and for $x \ne 0$, $f(x)$ is a composition of two continuous functions $g(x) = \mathrm{e}^x$ and $h(x) = -\frac{1}{x^2}$, that is, $f(x) = (g\circ h)(x)$.

(b) We will show that: For $x \ne 0$, $f^{(n)}(x) = P_n(y)\mathrm{e}^{-y^2}$, where $y = \frac{1}{x}$, and $P_n(y)$ is a polynomial of y with degree $3n$.

(1) When $n = 1$, we compute

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-y^2}(-2y)\cdot(-y^2) = 2y^3\mathrm{e}^{-y^2} = P_1(y)\mathrm{e}^{-y^2},$$

where $P_1(y) = 2y^3$ is a polynomial of y with degree 3.

(2) Assume that it is true for $n = k$, that is, $f^{(k)}(x) = \frac{\mathrm{d}^kf}{\mathrm{d}x^k} = P_k(y)\mathrm{e}^{-y^2}$, where $P_k(y)$ is a polynomial with degree $3k$.

(3) When $n = k+1$, we compute

$$\begin{aligned} f^{(k+1)}(x) = \frac{\mathrm{d}^{k+1}f}{\mathrm{d}x^{k+1}} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}^kf}{\mathrm{d}x^k} = \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\mathrm{d}^kf}{\mathrm{d}x^k}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}y}\left(P_k(y)\mathrm{e}^{-y^2}\right)(-y^2) \\ &= \left(\frac{\mathrm{d}P_k(y)}{\mathrm{d}y}\mathrm{e}^{-y^2} + P_k(y)\mathrm{e}^{-y^2}(-2y)\right)(-y^2) \\ &= \left(-y^2\frac{\mathrm{d}P_k(y)}{\mathrm{d}y} + 2y^3P_k(y)\right)\mathrm{e}^{-y^2}. \end{aligned}$$

Let $P_{k+1}(y) = -y^2\frac{\mathrm{d}P_k(y)}{\mathrm{d}y} + 2y^3P_k(y)$, which is a polynomial of y with degree $3+3k = 3(k+1)$.

(4) By mathematical induction, we know that for $x \neq 0$, $f^{(n)}(x) = P_n(y)\mathrm{e}^{-y^2}$, where $y = \frac{1}{x}$, and $P_n(y)$ is a polynomial of y with degree $3n$.

(c) Now, we will show that $\underline{f^{(n)}(0) = 0 \text{ for all } n \in \mathbb{N}}$.

(1) When $n = 1$, we compute

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\mathrm{e}^{-\frac{1}{x^2}}}{x} = \lim_{y\to\pm\infty} \frac{\mathrm{e}^{-y^2}}{\frac{1}{y}}$$
$$= \lim_{y\to\pm\infty} \frac{y}{\mathrm{e}^{y^2}} \overset{(\frac{\infty}{\infty}),L'}{=\!=} \lim_{y\to\pm\infty} \frac{1}{2y\mathrm{e}^{y^2}} = 0.$$

(2) Assume that it is true for $n = k$, that is, $f^{(k)}(0) = 0$.

(3) When $n = k + 1$, we compute

$$f^{(k+1)}(0) = \lim_{x\to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0} = \lim_{x\to 0} \frac{f^{(k)}(x)}{x} = \lim_{y\to\pm\infty} \frac{P_k(y)\mathrm{e}^{-y^2}}{\frac{1}{y}}$$
$$= \lim_{y\to\pm\infty} \frac{yP_k(y)}{\mathrm{e}^{y^2}} = 0.$$

Remark that we can apply l' Hospital Rule $\left[\frac{3n-1}{2}\right]$ times to get the limit is 0.

(4) By mathematical induction, we know that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

(d) Since $f(0) = 0$ and $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$, the Maclaurin series of $f(x)$ is

$$M(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots = 0.$$

This is a zero function, so the interval of convergence of $M(x)$ is $\mathbb{R}$. We compute the remainder

$$r_n(x) = f(x) - T_n(x) = f(x).$$

We get for any $x \neq 0$, $\lim_{n\to\infty} r_n(x) = \mathrm{e}^{-\frac{1}{x^2}} \neq 0$. Therefore, $f(x)$ is not equal to its Maclaurin series.

(e) For any integer $k \geq 0$, let $C^k(\mathbb{R})$ be the set (in fact, it is a vector space) consisting of all functions $f(x)$ that the derivatives $f'(x), f''(x), \ldots, f^{(k)}(x)$ exist and are continuous on $\mathbb{R}$. So $C^0(\mathbb{R})$, which is also denoted by $C(\mathbb{R})$, consists of all continuous functions on $\mathbb{R}$, and $C^\infty(\mathbb{R}) = \cap_{k=0}^{\infty} C^k(\Omega)$ consists of all smooth functions (continuous derivatives of all orders) on $\mathbb{R}$ (光滑函數).

Denote $C^\omega(\mathbb{R})$ be the set consisting of all smooth functions $f(x)$ that for all $x \in \mathbb{R}$, there exists $R > 0$ such that $f(x)$ equals its Taylor series expansion on $(x - R, x + R)$. We say a function $f(x) \in C^\omega(\mathbb{R})$ is *analytic* (解析函數).

這裡引進集合符號 $C^k(\mathbb{R})$, 表示 k 次求導之後仍連續的函數所成之集合。而 $C^\infty(\mathbb{R})$ 的元素是不論微分幾次函數都連續。至於能夠用泰勒級數重新表示的函數稱爲解析函數, 集合以 $C^\omega(\mathbb{R})$ 表示。

這個例子告知：存在光滑函數並非解析函數。

(f) The above discussion shows that the function $f(x)$ is a smooth function, but not an analytic function because $f(x)$ is not analytic at $x = 0$. So the conclusion is $C^{\omega}(\mathbb{R}) \subsetneq C^{\infty}(\mathbb{R})$.

Remark that we have the following relations:

$$C^{\omega}(\mathbb{R}) \subsetneq C^{\infty}(\mathbb{R}) \cdots \subsetneq C^{2}(\mathbb{R}) \subsetneq C^{1}(\mathbb{R}) \subsetneq C^{0}(\mathbb{R}).$$

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Example 6. Recall that the binomial series is

$$\sum_{n=0}^{\infty} C_n^m x^n = \sum_{n=0}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} x^n.$$

這裡要討論的是二項式級數在端點的收斂性。在 $m \leq 0$ 時可用之前所學的理論處理。當 $m > 0$ 需要額外更精細的討論。

We will check the convergence of the binomial series at the endpoints.

(a) If $m \leq -1$, then

$$\begin{aligned} |C_n^m x^n| = |C_n^m (\pm 1)^n| = |C_n^m| &= \left| \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} \right| \\ &= \frac{|m||(m-1)||(m-2)|\cdots|(m-n+1)|}{n!} \geq \frac{1\cdot 2\cdot 3\cdots n}{n!} = 1. \end{aligned}$$

So the series $\sum\limits_{n=0}^{\infty} C_n^m x^n$ is divergent at $x = \pm 1$ by the Test of Divergence.

(b_{-1}) If $-1 < m < 0$ and $x = -1$, then $0 < -m < 1$, and

$$\begin{aligned} C_n^m x^n &= \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}(-1)^n \\ &= \frac{(-m)(1-m)(2-m)\cdots(n-1-m)}{n!} \\ &= \frac{(-m)}{n}\cdot\frac{(1-m)}{1}\cdot\frac{(2-m)}{2}\cdots\frac{(n-1-m)}{n-1} \geq \frac{(-m)}{n}. \end{aligned}$$

Since $\sum\limits_{n=1}^{\infty} \frac{(-m)}{n}$ is divergent (p-series, $p = 1$), $\sum\limits_{n=0}^{\infty} C_n^m x^n$ is divergent at $x = -1$ by the Comparison Test.

(b_1) If $-1 < m < 0$ and $x = 1$, then $\sum\limits_{n=0}^{\infty} C_n^m x^n = \sum\limits_{n=0}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}$ is an alternating series. We compute

$$\begin{aligned} |C_n^m| &= \left| \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} \right| \\ &\geq \left| \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} \right| \left| \frac{m-n}{n+1} \right| = |C_{n+1}^m|, \end{aligned}$$

so it is a decreasing sequence. Next, we calculate

$$\begin{aligned} |C_n^m| &= \left| \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!} \right| \\ &= \left| \frac{m}{1}\cdot\frac{(m-1)}{2}\cdot\frac{(m-2)}{3}\cdots\frac{(m-n+1)}{n} \right| \\ &= \left| \left(1 - \frac{m+1}{1}\right)\left(1 - \frac{m+1}{2}\right)\cdots\left(1 - \frac{m+1}{n}\right) \right| = \prod_{k=1}^{n}\left(1 - \frac{m+1}{k}\right). \end{aligned}$$

Since

$$\ln|C_n^m| = \ln\left(\prod_{k=1}^{n}\left(1-\frac{m+1}{k}\right)\right) = \sum_{k=1}^{n}\ln\left(1-\frac{m+1}{k}\right) < \sum_{k=1}^{n} -\frac{m+1}{k}$$
$$= -(m+1)\sum_{k=1}^{n}\frac{1}{k}$$

and $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{k} = \sum\limits_{n=1}^{\infty}\frac{1}{n} = \infty$, we get

$$\ln\left(\lim_{n\to\infty}|C_n^m|\right) = \lim_{n\to\infty}\ln|C_n^m| = -\infty \Rightarrow \lim_{n\to\infty}|C_n^m| = 0.$$

By the Alternating Series Test, $\sum\limits_{n=0}^{\infty} C_n^m x^n$ is convergent.

(c) Before we check the case $m > 0$, we introduce the Raabe's Test:

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當 $m > 0$ 時, 二項式函數端點的收斂性需要用到另外的判別法則, 在此介紹 Raabe 判別法。

The Raabe's Test. *Suppose a series* $\sum\limits_{n=1}^{\infty} a_n$ *satisfies*

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \quad \textit{and} \quad \lim_{n\to\infty} n\left(\left|\frac{a_{n+1}}{a_n}\right| - 1\right) < -1,$$

then the series is absolutely convergent.

Remark that the p-series $\sum\limits_{n=1}^{\infty}\frac{1}{n^p}$ satisfies the condition, so the Raabe's Test is a Comparison Test with p-series.

<u>If $m > 0$</u>, then

$$\lim_{n\to\infty} n\left(\left|\frac{a_{n+1}}{a_n}\right| - 1\right) = \lim_{n\to\infty} n\left(\left|\frac{C_{n+1}^m}{C_n^m}\right| - 1\right) = \lim_{n\to\infty} n\left(\frac{\left|\frac{m(m-1)\cdots(m-n)}{n!}\right|}{\left|\frac{m(m-1)\cdots(m-n+1)}{n!}\right|} - 1\right)$$
$$= \lim_{n\to\infty} n\left(\frac{|m-n|}{n+1} - 1\right) = \lim_{n\to\infty} n\left(\frac{n-m}{n+1} - 1\right)$$
$$= -(m+1)\lim_{n\to\infty}\left(\frac{n}{n+1}\right) = -(m+1) < -1.$$

By the Raabe's Test, $\sum\limits_{n=0}^{\infty} C_n^m x^n$ is convergent.

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這裡提供二項式函數及其泰勒級數的相等之證明。注意到這邊不是透過驗證餘項趨近於零的方式，而是用微分方程的方法處理。

Example 7. We will prove $(1+x)^m = \sum_{n=0}^{\infty} C_n^m x^n$ on $|x|<1$.

(a) Let $g(x) = \sum_{n=0}^{\infty} C_n^m x^n$ on the interval of convergence $(-1,1)$. We will show that $\underline{(1+x)g'(x) = mg(x)}$ on the interval of convergence $(-1,1)$.

We compute $g'(x) = \sum_{n=1}^{\infty} C_n^m n x^{n-1}$ on the interval of convergence $(-1,1)$, and

$$\begin{aligned}
(1+x)g'(x) &= (1+x)\sum_{n=1}^{\infty} C_n^m n x^{n-1} = \sum_{n=1}^{\infty} C_n^m n x^{n-1} + \sum_{n=1}^{\infty} C_n^m n x^n \\
&= \sum_{n=0}^{\infty} C_{n+1}^m (n+1) x^n + \sum_{n=0}^{\infty} C_n^m n x^n \\
&= \sum_{n=0}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)(m-n)(n+1)}{(n+1)!} x^n \\
&\quad + \sum_{n=0}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)n}{n!} x^n \\
&= \sum_{n=0}^{\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)((m-n)+n)}{n!} x^n \\
&= m\sum_{n=0}^{\infty} C_n^m x^n = mg(x).
\end{aligned}$$

(b) Solve the differential equation $(1+x)g'(x) = mg(x), g(0) = 1, |x|<1$. It is separable equation, so we have

$$\frac{g'(x)}{g(x)} = \frac{m}{1+x} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(\ln g(x)) = \frac{m}{1+x} \Rightarrow \ln g(x) = m\ln(1+x) + C.$$

Since $g(0)=1$, we know that $C=0$. Hence $\ln g(x) = m\ln(1+x) = \ln(1+x)^m$ and it implies $g(x) = \sum_{n=0}^{\infty} C_n^m x^n = (1+x)^m$ on $|x|<1$.