Chapter 10 Parametric Equations and Polar Coordinates

10.1 Curves defined by parametric equations (page 640)

Definition 1 (page 640)**.** Suppose that x and y are both given as functions of a third variable t (called a *parameter*, 參數) by the equations

$$x = f(t), \quad y = g(t),$$

(called *parameter equations*, 參數方程). Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a *parametric curve* (參數曲線).

Sometimes t can be realized as "time" and we can interpret $(x, y) = (f(t), g(t))$ as the position of a particle at time t, but in many cases, t does *not* necessarily represent time, it is just a variable.

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將曲線用參數式的方法表達是從歐拉那個時候開始的，起初是把 t 想成時間，而曲線視爲質點隨時間變化時運動的軌跡。到後期，數學上再做抽象化的結果，t 不見得要和時間對應，它就只是一個參數（實數軸上的一個變數）而已。

各位首先要學習的是把以前所學的基本曲線，像是直線或是圓錐曲線等改用參數式表達。

Example 2. How do we express the following curves by parametric equations?

Curve	Parametric Equation
Straight line passing through (x_0, y_0)	$\begin{cases} x = \\ y = \end{cases}$
Circle with center (x_0, y_0) and radius r	$\begin{cases} x = \\ y = \end{cases}$
Ellipse $\dfrac{(x-x_0)^2}{a^2} + \dfrac{(y-y_0)^2}{b^2} = 1$	$\begin{cases} x = \\ y = \end{cases}$
Parabola $(x-x_0)^2 = 4p(y-y_0)$	$\begin{cases} x = \\ y = \end{cases}$
Hyperbola $\dfrac{(x-x_0)^2}{a^2} - \dfrac{(y-y_0)^2}{b^2} = 1$	$\begin{cases} x = \\ y = \end{cases}$

Example 3. Compare the following parametric equations:

(a) $(x, y) = (\cos t, \sin t), 0 \le t \le 2\pi$.

(b) $(x, y) = (\cos t, \sin t), 0 \le t \le 4\pi$.

(c) $(x, y) = (\cos 2t, \sin 2t), 0 \le t \le \pi$.

(d) $(x, y) = (\sin t, \cos t), 0 \le t \le 2\pi$.

(e) $(x, y) = (\cos t, -\sin t), 0 \le t \le 2\pi$.

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這幾個例子雖然軌跡都是圓，但是不同的參數表達會有些微的差別，試以質點運動軌跡的想法徹底理解參數式的表示與圖形的關係。

Example 4 (page 642)**.** Sketch the curve $x = \sin t, y = \sin^2 t$.

Solution.

The Cycloid, page 643

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學習參數式最標準的模型是擺線，各位首先要會將擺線的參數式確實表達，之後會從擺線開始探討微積分相關的理論。

Example 5 (page 643)**.** The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a *cycloid* (擺線). See Figure 1.

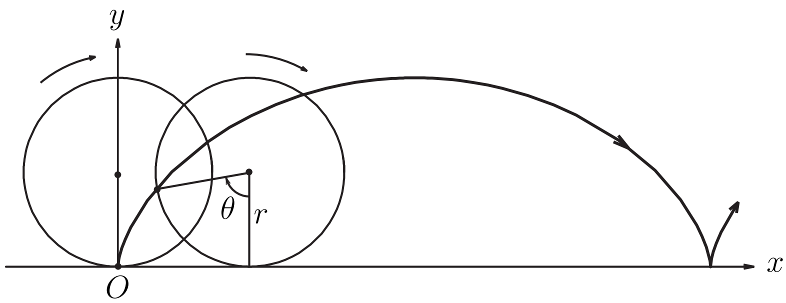

Figure 1: The cycloid.

If the circle has radius r and rolls along the x-axis and if one position of P is the origin, find parametric equations for the cycloid.

Solution.

There are many interesting problems related to cycloids.

Brachistochrone Problem (最速降線問題), page 644

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擺線在數學的發展上具有很重要的意義，如何製造一個溜滑梯軌道在磨擦力忽略不計的情況下質點從頂部沿軌道最快到達底部，稱爲最速降線問題，可用微積分的方法推出這個軌道是擺線。

Find the curve along which a particle will slide in the shortest time (under the influence of gravity) from a point A to a lower point B not directly beneath A.

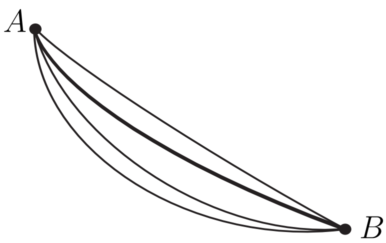

Figure 2: Brachistochrone Problem.

The Swiss mathematician John Bernoulli, who posed this problem in 1696, showed that among all possible curves that join A to B, the particle will take the least time sliding from A to B if the curve is part of an inverted arch of a cycloid.

Tautochrone Problem (等時降線), page 644

The Dutch physicist Huygens had already shown that the cycloid is also the solution to the tautochrone problem: no matter where a particle P is placed on an inverted cycloid, it takes the same time to slide to the bottom.

擺線的另一個重要意義是等時降線: 在擺線上任意一處放置球, 而球沿擺線的軌道滾動至底部所花的時間相同。

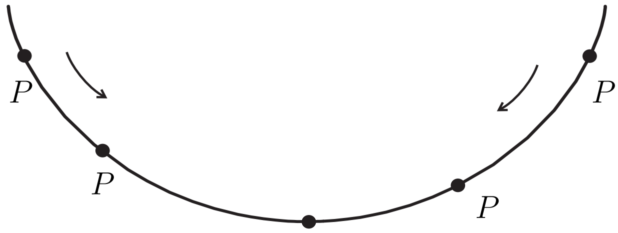

Figure 3: Tautochrone Problem.

Huygens proposed that pendulum clocks should swing in cycloidal arcs because then the pendulum would take the same time to make a complete oscillation whether it swings through a wide or a small arc.

Graphing Devices, page 644

We can use graphing devices to sketch complicated curves. The curves shown in Figure 4 are almost impossible to produce by hand.

各位小時候應該有玩過繁花規, 就是一枝筆插在一個圓形的洞洞板中繞著另一個較大的圓形滾動就會畫出像是左圖的曲線。你會看到左圖的三條曲線參數式都是由正弦函數與餘弦函數組成, 但是振幅與頻率的不同就會造出曲線的多樣性。

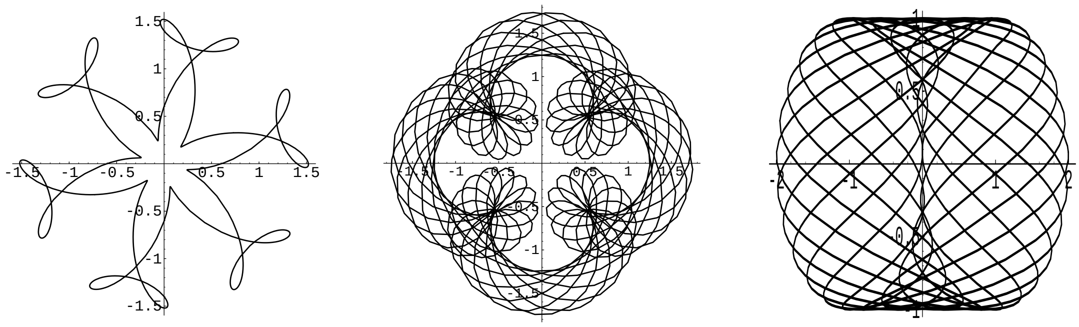

Figure 4: (a) $x = \sin t + \frac{1}{2}\cos 5t + \frac{1}{4}\sin 13t, y = \cos t + \frac{1}{2}\sin 5t + \frac{1}{4}\cos 13t, t \in [0, 2\pi]$.
(b) $x = \sin t + \frac{1}{2}\sin 5t + \frac{1}{4}\cos 2.3t, y = \cos t + \frac{1}{2}\cos 5t + \frac{1}{4}\sin 2.3t, t \in [0, 20\pi]$.
(c) $x = \sin t - \sin 2.3t, y = \cos t, t \in [0, 20\pi]$.

10.2 Calculus with Parametric Curves (page 649)

We now apply the methods of calculus to parametric curves. In particular, we solve problem involving tangents, area, arc length, surface area, and volume.

Tangents, page 649

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Suppose f and g are differentiable functions and we want to find the tangent line at a point on the curve $x = f(t), y = g(t)$, where y is also a differentiable function of x. Then the Chain Rule gives

這裡要了解的是當曲線以參數式表達時，$y'(x)$ 或切線斜率的找法。注意到 $y''(x)$ 的結果較為複雜。

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \Rightarrow \underline{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0.}$$

We can compute the second derivative $\frac{d^2y}{dx^2}$ as follows:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)}{\frac{dx}{dt}} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

☐ 特別注意: $\frac{d^2y}{dx^2} \neq \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$。

以擺線為例，計算在 $\theta = \frac{\pi}{3}$ 處的切線方程式。並觀察擺線具有水平切線及鉛直切線的所在位置。

Example 1 (page 650)**.**

(a) Find the tangent line to the cycloid $x = r(\theta - \sin\theta), y = r(1 - \cos\theta)$ at the point where $\theta = \frac{\pi}{3}$.

(b) At what points its tangents horizontal? When is it vertical?

Solution.

Areas, page 651

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\,\mathrm{d}x$,where $F(x) \geq 0$. If the curve is traced out once by the parameter equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

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因爲函數的積分對應到的幾何意義是有方向有符號的面積，所以當曲線用參數式表達時，若要呈現面積大小時，必須和圖形對應，上限與下限的對應遵照其幾何意義決定。

$$A = \int_a^b y\,\mathrm{d}x = \int_\alpha^\beta g(t)f'(t)\,\mathrm{d}t \quad \text{or} \quad \int_\beta^\alpha g(t)f'(t)\,\mathrm{d}t.$$

Example 2 (page 651)**.** Find the area under one arch of the cycloid $x = r(\theta - \sin\theta), y = r(1 - \cos\theta)$.

Solution.

Arc Length, page 652

We already know how to find the length L of a curve C given in the form $y = F(x), a \leq x \leq b$. If $F'(x)$ is continuous, then

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至於曲線用參數式表達若要算曲線長，因爲積分的函數恆正，所以積分下限與上限就是由小到大，得到的結果也會是正的值。

$$L = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Suppose that C can also be described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, where $\frac{\mathrm{d}x}{\mathrm{d}t} = f'(t) > 0$. This means that C is traversed once, from left to right, as t increases from α to β and $f(\alpha) = a, f(\beta) = b$. Then we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_\alpha^\beta \sqrt{1 + \left(\frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}\right)^2}\,\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \int_\alpha^\beta \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

The above formula is generally true even if C can't expressed in the form $y = F(x)$.

Theorem 3 (page 649)**.** *If a curve C is described by the parametric equations $x = f(t), y = g(t), \alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is*

$$L = \int_\alpha^\beta \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

在看影片之前可先猜一猜擺線一拱的弧長，然後再透過影片的學習確定你的直覺與結果是否一致。

Example 4 (page 653)**.** Find the length of one arch of the cycloid $x = r(\theta - \sin\theta), y = r(1 - \cos\theta)$.

Solution.

Surface Area, page 654

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若曲線用參數式表達，要計算旋轉體體積時， 也是按照第八章的概念列式，積分的下限與上限也是由小到大(遵照曲線弧長的概念而得)。

In the same way as for arc length, we can obtain a formula for surface area. If the curve given by the parametric equations $x = f(t), y = g(t), \alpha \le t \le \beta$, is rotated about the x-axis, where f', g' are continuous and $g(t) \ge 0$, then the area of the resulting surface is given by

$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

The general symbolic formulas $S = \int 2\pi y\,\mathrm{d}s$ and $S = \int 2\pi x\,\mathrm{d}s$ are still valid, but for parametric curves we use

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

習學如何用微積分計算球的表面積。

Example 5 (page 654)**.** Show that the surface area of a sphere of radius r is $4\pi r^2$.

Solution.

Volume

See section 6.2 and 6.3.

10.3 Polar Coordinates (page 658)

A coordinate system represents a point in the plane by an ordered pair of numbers. We usually use Cartesian coordinates (笛卡爾坐標, 直角坐標), which are directed distances from two perpendicular axes. Here we describe another coordinate system introduced by Newton, called the *polar coordinate system* (極坐標).

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認識極坐標。平面中的點之所在位置有多種方法呈現，平常較爲熟悉的是直角坐標，而極坐標是改用極點到點的有向距離 r 以及廣義角 θ 這兩個數字表達點的位置。注意到這裡的 r 可正可負，而極坐標是一個多值的對應關係。

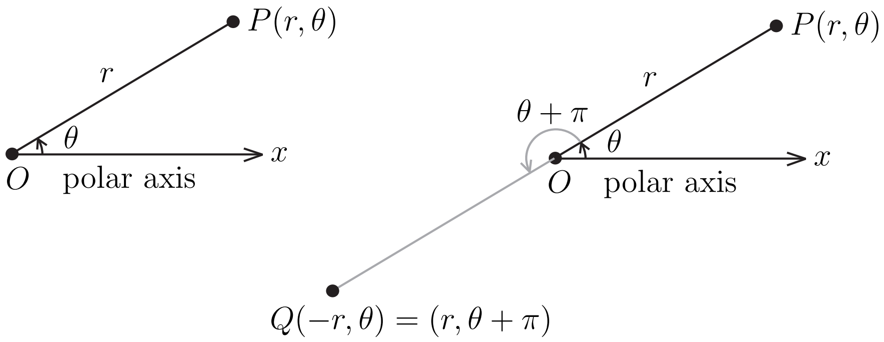

Figure 1: Polar coordinate system. We choose a point in the plane that is called the *pole* and is labeled O. Then we draw a ray starting at O called *polar axis*, which is usually corresponds to the positive x-axis in Cartesian coordinates.

Here are some remarks about the polar coordinate system.

- If P is any other point in the plane, let r be the distance from O to P and let θ be the angle between the polar axis and the line OP. Then the point P is represented by the ordered pair (r,θ) are called *polar coordinates* of P.
- We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P=O$, then $r=0$ and we agree that $(0,\theta)$ represents the pole for any value of θ.
- The points $(-r,\theta)$ an (r,θ) lie on the same line through O and at the same direction $|r|$ from O, but on opposite sides of O. If $r>0$, the point (r,θ) lies in the same quadrant as θ; if $r<0$, it lies in the quadrant on the opposite side of the pole.
- Notice that $(-r,\theta)$ represents the same point as $(r,\theta+\pi)$.
- The connection between polar and Cartesian coordinates:

直角坐標與極坐標的轉換關係必須確實了解。

  (a) $\cos\theta=\dfrac{x}{r}, \quad \sin\theta=\dfrac{y}{r}.$ (角度和直角坐標與半徑的關係)

  (b) $x=r\cos\theta, \quad y=r\sin\theta.$ (直角坐標用極坐標表達)

  (c) $r^2=x^2+y^2, \quad \tan\theta=\dfrac{y}{x}.$ (極坐標可以用直角坐標表達)

Polar Curves, page 660

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The *graph of a polar equation* (極坐標方程式) $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one representation (r, θ) whose coordinates satisfy the equation.

熟悉一些極坐標方程式對應的圖形。有些極坐標方程式表達看似簡單，但是圖形卻不是那麼好想像，可利用數學繪圖軟體多多體會。

Example 1 (page 660-662)**.** Plot the following curves represented by the polar equation.

(a) $r = 2$. ____________________

(b) $\theta = \frac{\pi}{4}$. ____________________

(c) $r = 2\cos\theta$. ____________________

(d) $r = 1 + \sin\theta$. ____________________

(e) $r = \cos 2\theta$. ____________________

Solution.

Symmetry, page 663

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When we sketch polar curves it is sometimes helpful to take advantage of symmetry.

善用圖形的對稱性還有方程式的平移縮放旋轉理論可以幫助我們把一些曲線做連結。特別是 (d) 的部份我覺得很值得仔細思考。

(a) If a polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about ____________________.

(b) If the equation is unchanged when r is replaced by $-r$, or when θ is replaced by $\theta + \pi$, the curve is symmetric about ____________________.

(c) If the equation is unchanged when θ is replaced by $\pi - \theta$, the curve is symmetric about ____________________.

(d) Given $r = f(\theta)$, consider the curve $r = af(b\theta + c) + d$, where $a, b, c, d, \in \mathbb{R}$.

Tangents to Polar Curves, page 663

To find a tangent line to a polar curve $r = f(\theta)$, we regard θ as a parameter and write its parametric equations as

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當曲線用極坐標方程式表達的時候, 把 θ 想成是參數, 再用直角坐標與極坐標的轉換關係, 可以得到曲線的參數式表達, 這麼一來就可以用參數式的概念計算 $y'(x)$。

$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta. \end{cases}$$

Using the method for finding slopes of parametric curves and the Product Rule, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \underline{\hspace{6cm}} \tag{1}$$

- Horizontal tangents: ________________________ (provided that $\frac{\mathrm{d}x}{\mathrm{d}\theta} \neq 0$).
- Vertical tangents: ________________________ (provided that $\frac{\mathrm{d}y}{\mathrm{d}\theta} \neq 0$).
- Tangent lines at the pole: we put $r = 0$ into formula (1) and get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \underline{\hspace{6cm}}$$

Example 2 (page 664)**.**

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心臟線是研究極坐標方程式的標準模型, 藉此熟悉極坐標並推得其切線方程。

(a) For the cardioid $r = 1 + \sin\theta$, find the slope of the tangent line when $\theta = \frac{\pi}{3}$.

(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

Solution.

Graphing Polar Curves with graphing Devices

我們可用數學繪圖軟體幫助了解極坐標方程式的圖形，下方有幾個很有特色的曲線，各位不妨將方程式輸入到數學繪圖軟體看看它的長相。

We can use graphing devices, Desmos Calculator for example, to sketch complicated curves. The curves shown in Figure 2 are almost impossible to produce by hand.

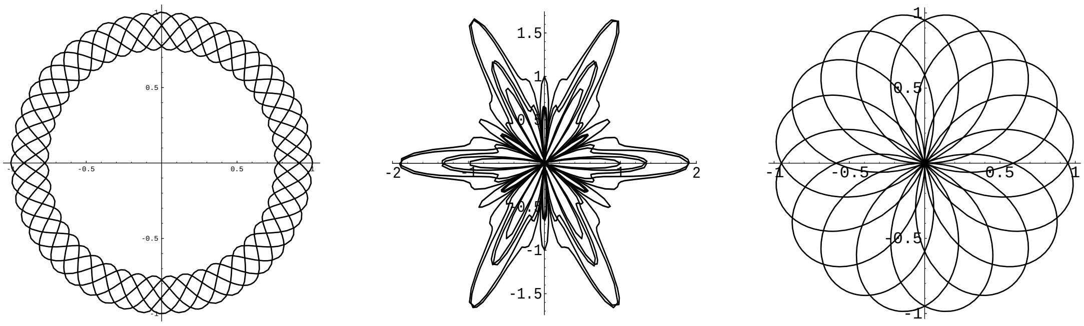

Figure 2: (a) $r = \sin^2(2.4\theta) + \cos^4(2.4\theta), \theta \in [\frac{-2\pi}{2.4}, \frac{22\pi}{2.4}]$. (b) $r = \sin^2(1.2\theta) + \cos^3(6\theta), \theta \in [0, 6\pi]$ (c) $r = \sin\left(\frac{8}{5}\theta\right), \theta \in [0, 10\pi]$.

Some interesting curves and their polar equations.

(a) $r = a\sin(b\theta)$: rose or rhodonea curve (玫瑰線).

(b) $r = a + b\theta$: Archimedean spiral (阿基米德螺線; 等速螺線).

(c) $r = ae^{b\theta}$: logarithmic spiral (對數螺線).

(d) $r^2 = \sin 2\theta$: lemniscate (雙紐線).

(e) $r = e^{\sin\theta} - 2\cos(4\theta)$: butterfly curve (蝶形線).

(f) $r = 1 + c\sin\theta$: limacons de Pascal. (帕斯卡蝸線).

(g) $r = 1 + 2\sin(\frac{\theta}{2})$: nephroid of Freeth.

(h) $r = \sqrt{1 - 0.8\sin^2\theta}$: hippopede.

(i) $r = |\tan\theta|^{|\cot\theta|}$: Valentine curve.

10.4 Areas and Length in Polar Coordinates (page 669)

In this section, we develop the formula for the area of the region whose boundary is given by a polar equation.

Example 1 (page 669)**.** The area of a sector of a circle with the radius r and the radian θ is $A =$ ________.

xNsXx2N01hA

Example 2 (page 669)**.** Find the area of a region $\mathcal{R}$ bounded by the polar curve $r = r(\theta)$ and by rays $\theta = a$ and $\theta = b$, where $r(\theta)$ is a positive continuous function and $0 < b - a \leq 2\pi$.

我們可以用極坐標方程式計算區域面積， 若是對角度進行分割，再取樣本點， 透過扇形的面積加總後取極限，也可以得到面積的積分公式。

Solution.

Example 3 (page 670)**.** Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Solution.

Example 4 (page 671)**.** Find the area of the region $\mathcal{R}$ bounded by curves with polar equations $r = f(\theta), r = g(\theta), \theta = a$, and $\theta = b$, where $f(\theta) \geq g(\theta) \geq 0$ and $0 < b - a \leq 2\pi$.

更一般地，當區域是由兩個極坐標方程式圍住時， 其區域面積也可以寫出，基本上就是大扇形面積減掉小扇形面積的概念。

Solution.

To find *all* points of intersection of two polar curves, it is recommended that you draw the graphs of both curves.

4XFn7sacQDY

Example 5 (page 671)**.** Find all points of intersection of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.

Solution.

兩個極坐標方程式的交點實際上是一個非常困難的問題，這是因爲極坐標是多值函數。現在我們有數學繪圖軟體的輔助，可以幫助我們更容易了解交點的所在位置。

Arc Length, page 671

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To find the length of a polar curve $r = r(\theta), a \le \theta \le b$, we regard θ as a parameter and write the parameter equations of the curves as

曲線弧長的計算方式，也是將問題轉變成參數式之後再進行計算。

$$\begin{cases} x = r\cos\theta = r(\theta)\cos\theta \\ y = r\sin\theta = r(\theta)\sin\theta. \end{cases}$$

Then using the Product Rule and differentiating with respect to θ, we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} =$$
$$\frac{\mathrm{d}y}{\mathrm{d}\theta} =$$

so

$$\begin{aligned}\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 &= (r'(\theta))^2\cos^2\theta - 2r\cdot(r'(\theta))^2\cos\theta\sin\theta + r^2\sin^2\theta \\ &\quad + (r'(\theta))^2\sin^2\theta + 2r\cdot(r'(\theta))^2\sin\theta\cos\theta + r^2\cos^2\theta \\ &= \end{aligned}$$

Assuming that $f'(\theta)$ is continuous, we can write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta =$$

Example 6 (page 672)**.** Find the length of the cardioid $r = 1 + \sin\theta$.

看影片前不妨先猜一猜心臟線的弧長，再透過影片的解說對照你的直覺與結果是否一致。

Solution.

Surface Area, page 674

The area of the surface generated by rotating the polar curve $r = f(\theta), a \le \theta \le b$ (where $f'(\theta)$ is continuous and $0 \le a < b \le \pi$) about the polar axis is

9Ex-VwbwLRI

極坐標方程式的旋轉體表面積公式，也是將問題轉化爲參數式之後再重現。

Surface area =

The area of the surface generated by rotating the polar curve $r = f(\theta), a \le \theta \le b$ (where $f'(\theta)$ is continuous and $0 \le a < b \le \pi$) about the line $\theta = \frac{\pi}{2}$ is

Surface area =

Volume

See section 6.2 and 6.3.