Chapter 8 Further Applications of Integration

8.1 Arc Length (page 544)

The Arc Length Formula (page 544)**.** *If f' is continuous on $[a,b]$, then the length of the curve $y = f(x), a \le x \le b$, is*

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x. \tag{1}$$

If a curve has the equation $x = g(y), c \le y \le d$, and $g'(y)$ is continuous, then by interchanging the roles of x and y, we obtain the following formula for its length:

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\,\mathrm{d}y = \int_c^d \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y. \tag{2}$$

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認識曲線長度的積分公式，在分割樣本點取和求極限的過程中， 所有分割點都必須選在曲線上， 而每一小段的曲線弧長利用線段長估計。透過均值定理，線段長可以和當中某一點的切線斜率產生連繫，就把那一點當成樣本點。最後得到的積分函數，本質上是切向量的長度，會在本單元的最後補充說明。

Proof. Suppose that a curve C is defined by the equation $y = f(x)$, where $f(x)$ is continuous and $a \le x \le b$. We obtain a polygonal approximation to C by dividing the interval $[a,b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx. If $y_i = f(x_i)$, then the point $P_i(x_i, y_i)$ lies on C and the polygon with vertices $P_0, P_1, \ldots, P_n$ is an approximation to C.

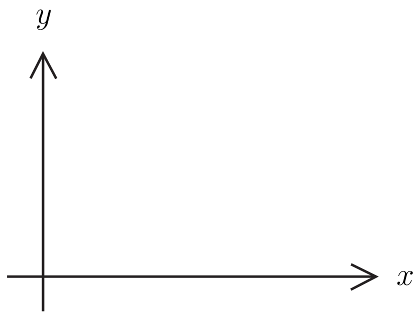

Figure 1: We use the length of inscribed polygons to approximate the length of C.

We define the *length* L (弧長) of the curve C with equation $y = f(x), a \le x \le b$, as the limit of the lengths of these inscribed polygons (if the limit exists): $L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$.

When f' is continuous on $[a,b]$ (we say $f \in C^1([a,b])$), then by the Mean Value Theorem, there is a number $x_i^* \in (x_{i-1}, x_i)$ such that

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (f(x_i) - f(x_{i-1}))^2} \\ &= \sqrt{(\Delta x)^2 + (f'(x_i^*)\Delta x)^2} = \sqrt{1 + (f'(x_i^*))^2}\Delta x. \end{aligned}$$

Therefore,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| =$$

To get the formula (2), by similar discussion, we divide $c \le y \le d$ into n subinterval with endpoints $y_0, y_1, \ldots, y_n$ and equal width Δy, and rewrite $|P_{i-1}P_i|$ as

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(g(y_i) - g(y_{i-1}))^2 + (\Delta y)^2} \\ &= \sqrt{(g'(y_i^*)\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + (g'(y_i^*))^2}\Delta y, \end{aligned}$$

so

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| =$$

□

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Example 1. Show that the circumference of a circle with radius r is $2\pi r$.

Solution.

若是把上半圓與下半圓分別看成是函數的圖形，計算弧長時要處理的積分會是瑕積分，這是因爲兩端點的切線斜率會是正負無限大的關係。

Example 2. Find the length of the curve $x = \frac{1}{3}\sqrt{y}(y-3), 1 \le y \le 9$.

Solution.

這條曲線可以確實算出弧長，是因爲根號內部又可以整理成完全平方的形式，就可以和根號去掉。各位平時應該要一直記住：平方再開根號要加絕對值，所以平時應養成習慣，確定平方與根號去掉後的量是正量。

這個例題另一方面示範以 y 爲變數的方式計算曲線弧長。

Example 3 (page 549)**.** Find the length of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

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各位在看影片前，應該先猜一猜：你覺得星形線的長度會比圓周長大還是小？然後再看影片以確定你的直覺是否正確。

Solution.

The Arc Length Function

We will find it useful to have a function that measures the arc length of a curve from a particular starting point to any other point on the curve. If a smooth curve C has the equation $y = f(x), a \le x \le b$, let $s(x)$ be the distance along C from the initial point $P_0(a, f(x))$ to the point $Q(x, f(x))$. Then

弧長函數應細細體會，這個函數就是度量從某個點爲起始點之下，到對應點之間的弧長。

弧長函數的變化率是該點的切向量長度，這是微積分基本定理順勢得出的結果。

$$s(x) = \int_a^x \sqrt{1 + (f'(t))^2}\,\mathrm{d}t$$

is a function, called the *arc length function* (弧長函數).

By the Fundamental Theorem of Calculus, we get

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + (f'(x))^2} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$$

In differential sense, we can view the arc length as the infinitesimal Pythagorean theorem: $(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2$.

Similarly, we have

$$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y.$$

Remark 4. In general, we can viewed the curve as $\mathbf{r}(x) = (x, f(x))$. Then we have $\mathbf{r}'(x) = (1, f'(x))$ and $|\mathbf{r}'(x)| = \sqrt{1 + (f'(x))^2}$, so the arc-length formula becomes

$$s(x) = \int_a^x |\mathbf{r}'(t)|\,\mathrm{d}t.$$

8.2 Area of a Surface of Revolution (page 551)

In this section, we will derive the formula of the area of a surface of revolution.

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理解圓柱表面積、圓錐表面積、錐台表面積的推導。學習這些幾何物件的表面積之目的是在下一個影片中要推導旋轉體表面積公式。

Example 1 (page 551)**.** Find the lateral surface area of a circular cylinder with radius r and height h.

Solution.

Example 2 (page 551)**.** Find the lateral surface area of a circular cone with base radius r and slant height l.

Solution.

Example 3 (page 551)**.** Find the lateral surface area of a *band*, which is a portion of a circular cone with upper radius r_1, lower radius r_2, and slant height l.

Solution.

Example 4 (page 552)**.** Find the surface area of the surface obtained by rotating the curve $y = f(x), a \le x \le b$, about the x-axis. (Assume that $f \in C^1([a, b])$.)

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Solution.

(a) Partition: $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. We have $\Delta x = x_i - x_{i-1} = \frac{b-a}{n}$.

(b) Approximate the curve $(x, f(x))$ by polygons $\cup_{i=1}^{n} \overline{P_{i-1}P_i}$, where $P_i = (x_i, f(x_i))$.

(c) By the mean value theorem, the surface area is approximated by

$$\begin{aligned} S_n &= \sum_{i=1}^{n} 2\pi \left(\frac{f(x_{i-1}) + f(x_i)}{2} \right) |\overline{P_{i-1}P_i}| \\ &= \sum_{i=1}^{n} 2\pi \left(\frac{f(x_{i-1}) + f(x_i)}{2} \right) \sqrt{(f(x_i) - f(x_{i-1}))^2 + (x_i - x_{i-1})^2} \\ &= \end{aligned}$$

where $x_i^{**} \in [x_{i-1}, x_i]$ satisfies $f(x_i^{**}) = \frac{f(x_{i-1})+f(x_i)}{2}$ and $x_i^* \in [x_{i-1}, x_i]$.

(d) When $n \to \infty$, we have

$$\begin{aligned} &\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \left(f(x_i^*) + (f(x_i^{**}) - f(x_i^*)) \right) \sqrt{1 + (f'(x_i^*))^2}\, \Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1 + (f'(x_i^*))^2}\, \Delta x + \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi (f(x_i^{**}) - f(x_i^*)) \sqrt{1 + (f'(x_i^*))^2}\, \Delta x \\ &= \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, \mathrm{d}x. \end{aligned}$$

Here we need to estimate (use mean value theorem and extreme value theorem)

$$\begin{aligned} &\sum_{i=1}^{n} 2\pi \left| f(x_i^{**}) - f(x_i^*) \right| \sqrt{1 + (f'(x_i^*))^2}\, \Delta x \\ &\le \sum_{i=1}^{n} 2\pi |f'(x_i^{***}) \Delta x| \sqrt{1 + (f'(x_i^*))^2}\, \Delta x \le 2\pi M \sqrt{1 + M^2} \sum_{i=1}^{n} (\Delta x)^2 \\ &= 2\pi M \sqrt{1 + M^2} \sum_{i=1}^{n} \left(\frac{b-a}{n^2} \right)^2 = 2\pi M \sqrt{1 + M^2} (b-a)^2 \cdot \frac{1}{n} \to 0 \end{aligned}$$

as $n \to \infty$.

各位應該花時間好好體會旋轉體表面積的公式推導，給定一條曲線，利用分割樣本點取和求極限的過程，所有分割點都必須選在曲線上，在每個小區間內用錐台的表面積估計旋轉體表面積，最後得到的公式在最下方。

這裡應強調的是證明的難點，在步驟 (c) 中，對於曲線弧長的部份，利用均值定理得到了樣本點 x_i^*，但是另一方面錐台表面積的公式中，是選用了中點 x_i^{**}，這是不同的點，所以不能直接寫出對應的積分式(回想定積分的定義，在黎曼和的階段，每一小段當中必要選到一模一樣的樣本點)。爲此，我們在 (c) 的最後一步驟，故意做 $f(x_i^*)$ 一加一減，則其中一部份可以湊出積分式；另一方面，必須再進一步估計剩下的量是高階的無窮小量，這段的論述就是 (d) 後面的估計。

Definition 5. *Surface area* (表面積) of the surface obtained by rotating the curve $y = f(x), a \le x \le b$, about x-axis is

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$$S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, \mathrm{d}x = \int_a^b 2\pi y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x} \right)^2}\, \mathrm{d}x.$$

旋轉體表面積公式：圓周長乘以弧長的積分。

Recall that the differential of the arc length function is

$$(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 \Rightarrow \mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y. \tag{3}$$

If the curve is described as $x = g(y), c \leq y \leq d$, then the formula for surface area (rotating about x-axis) becomes

$$S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y. \tag{4}$$

Formula (3) and (4) can be formally written as

$$\int 2\pi y\,\mathrm{d}s.$$

For rotation about the y-axis, the surface area formula (formally) becomes

$$S = \int 2\pi x\,\mathrm{d}s, \quad \text{where} \quad \mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$$

☐ 表面積公式: 圓周長乘以弧長的積分。

例題示範旋轉體表面積的計算。

Example 6. Find the area of surface obtained by rotating $y = \sin x, 0 \leq x \leq \pi$, about the x-axis.

Solution.

Example 7 (page 556)**.** Consider the region $\mathcal{R} = \{(x, y) | x \geq 1, 0 \leq y \leq \frac{1}{x}\}$ rotating about the x-axis.

(a) Show that the volume of the resulting solid is finite.

(b) Show that the surface area is infinite. (The surface is called *Gabriel's horn.*)

Solution.

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這個例子在數學上具有非常深刻的意義，一個體積有限的實體，它的表面積可以無限，也就是你無法拿油漆幫這個實心物體完全上色。這個例子衝擊到大家對於不同維度之間的一些直觀看法。