Chapter 7 Techniques of Integration

7.1 Integration by Parts, page 472

The rule that corresponds to the Product Rule for differentiation is called the rule for *integration by parts* (分佈積分).

FX9qNzLurjY

第七章前四節是在介紹積分技巧；而分佈積分法是探討兩類型函數相乘的積分處理法。你會看到這個公式的特色在於等式的左邊和右邊微分的函數不同。換言之，這個公式可以把微分轉移到另一個函數身上。

The Product Rule states that if f and g are differentiable functions, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x).$$

In the notation for indefinite integrals this equation becomes

$$\int (f(x)g'(x) + g(x)f'(x))\,\mathrm{d}x = f(x)g(x), \quad \text{or}$$
$$\int f(x)g'(x)\,\mathrm{d}x + \int g(x)f'(x)\,\mathrm{d}x = f(x)g(x).$$

So we can rearrange this equation as

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x. \tag{1}$$

接下來會仔細討論任兩類型函數的積分，到底該把誰看成 u，該把誰看成 v，這個方法是有邏輯可循的，所以重點是要把當中的邏輯學起來。

Formula (1) is called the *formula for integration by parts.*

Let $u = f(x)$ and $v = g(x)$, then the differentials are $\mathrm{d}u = f'(x)\,\mathrm{d}x$ and $\mathrm{d}v = g'(x)\,\mathrm{d}x$. By the Substitution Rule, the formula for integration by parts becomes

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u. \tag{2}$$

Example 1 (page 472)**.** Evaluate $\displaystyle\int x\sin x\,\mathrm{d}x$.

若遇到多項式與三角函數相乘的積分，因爲多項式微分之後會降次，降到變成常數之後就只剩下三角函數的積分；所以使用分佈積分時，先把三角函數積分後放到 d 的右邊，利用分佈積分就可以把多項式微分。

Solution.

Example 2 (page 473)**.** Evaluate $\displaystyle\int \ln x\,\mathrm{d}x$.

3RH0T-cTxFs

對數函數的積分，直接使用分佈積分法，它已經分配成 $\int u\,\mathrm{d}v$ 的型式了。

Solution.

若遇到多項式與對數相乘的積分，因爲 $\ln x$ 微分是 $\frac{1}{x}$，所以使用分佈積分時，先把多項式積分後放到 d 的右邊，分佈積分後就可以將對數函數微分。注意到，先把多項式積分將增加多項式的次方，似乎不利積分的處理，但是分佈積分後對數微分產生 $\frac{1}{x}$ 會把次數降一次，所以情況並沒有變糟。

Example 3. Find the integral $\displaystyle\int_1^{\mathrm{e}} x(\ln x)^2 \,\mathrm{d}x$.

Solution.

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Example 4. Evaluate $\displaystyle\int x\,\mathrm{e}^x \,\mathrm{d}x$.

Solution.

若遇到多項式與指數函數相乘的積分，因爲多項式微分之後會降次，降到變成常數之後就只剩下指數函數的積分；所以使用分佈積分時，先把指數函數積分後放到 d 的右邊，在分佈積分後，就可以把多項式微分。

若遇到指數函數與三角函數 $(\sin x, \cos x)$ 相乘的積分，因爲這兩類函數的微分都有週期性（和原函數相關），所以使用分佈積分的時候，這兩種函數都可以當成 u 或 v。建議兩種變換的方式都確實地操作一次以清楚了解該原理。

Example 5 (page 474)**.** Evaluate $\displaystyle\int \mathrm{e}^x \sin x \,\mathrm{d}x$.

Solution.

☐ $\sin x$ 和 $\cos x$ 的微分有週期性，利用兩次分佈積分之後再將整個等式處理。

Example 6 (page 475)**.** Evaluate $\int \tan^{-1} x \, \mathrm{d}x$.

Solution.

6Qqv9n1lUx4

反三角函數的積分，直接使用分佈積分法，它已經分配成 $\int u \, \mathrm{d}v$ 的型式了。

☐ 反三角函數的積分，直接用分佈積分法。

Example 7. Consider the region $\mathcal{R}$ enclosed by the curves $y = \cos x$ and $y = \sin x$, and $0 \le x \le \frac{\pi}{4}$. Find the volume of the solid obtained by rotating the region about the y-axis.

Solution.

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學過分佈積分法之後，就可以處理更多的應用問題，特別是用柱殼法計算旋轉體體積的時候，因爲公式本身會帶有 x，只要函數是不同於多項式的函數，確實處理積分時就會用到分佈積分法。

Example 8. Find the average of the horizontal chords in $y = \sin x, 0 \le x \le \pi$.

Solution.

FRFcI1UbVis

這個例題看似平凡無奇，但是它和著名的布豐投針 (Buffon's Needle) 相關，有興趣者可以繼續網路搜尋相關資訊。注意函數與水平弦相交時，x 的表示法要會寫。

7.2 Trigonometric Integrals, page 479

qPiDTZQdBWY

這個單元要學習的是三角積分，有兩種類型，第一類要處理的是 $\sin x$ 與 $\cos x$ 各種次方的相乘之積分。這裡雖然用非常一般的記號直接計算其結果，但是各位應該要理解其原理：只要次方有一者是奇數，那就把一個丟到（積分）d 的右手邊，剩下的偶數次方，透過三角恆等式 $\sin^2 x + \cos^2 x = 1$ 就可以變成以 $\sin x$ 或 $\cos x$ 爲變數的「多項式」積分。若兩者的次方都是偶數，那就用半角公式讓次方減半，不斷減半的情形下，終究會讓某個函數的次方變成奇數。

In this section we use trigonometric identities to integrate certain combinations of trigonometric functions.

Example 1 (page 481)**.** Evaluate $\int \sin^m x \cos^n x \,\mathrm{d}x$, where $m, n \geq 0$ are integers.

Solution.

(a) If $\underline{m = 2k+1}$, then

$$\int \sin^m x \cos^n x \,\mathrm{d}x = \int \sin^{2k+1} x \cos^n x \,\mathrm{d}x = \underline{\hspace{5cm}}$$

$$= -\int (1-\cos^2 x)^k \cos^n x \,\mathrm{d}\cos x = -\int \sum_{i=0}^{k} C_i^k 1^{k-i} (-1)^i \cos^{2i} x \cos^n x \,\mathrm{d}\cos x$$

$$= \sum_{i=0}^{k} (-1)^{i+1} C_i^k \int \cos^{n+2i} x \,\mathrm{d}(\cos x) = \underline{\hspace{6cm}}.$$

(b) If $\underline{n = 2k+1}$, then

$$\int \sin^m x \cos^n x \,\mathrm{d}x = \int \sin^m x \cos^{2k+1} x \,\mathrm{d}x = \underline{\hspace{4cm}}$$

$$= \int \sin^m x (1-\sin^2 x)^k \,\mathrm{d}\sin x = \int \sin^m x \sum_{i=0}^{k} C_i^k 1^{k-i} (-1)^i \sin^{2i} x \,\mathrm{d}\sin x$$

$$= \sum_{i=0}^{k} (-1)^i C_i^k \int \sin^{m+2i} x \,\mathrm{d}\sin x = \underline{\hspace{6cm}}.$$

(c) If $\underline{m = 2k, n = 2l}$, then using the half-angle identities

$$\sin^2 x = \underline{\hspace{2.5cm}} \quad \text{and} \quad \cos^2 x = \underline{\hspace{2.5cm}},$$

we have

$$\int \sin^m x \cos^n x \,\mathrm{d}x = \int \sin^{2k} x \cos^{2l} x \,\mathrm{d}x$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^k \left(\frac{1+\cos 2x}{2}\right)^l \mathrm{d}x = \sum_{i=0}^{k} \sum_{j=0}^{l} \frac{(-1)^i C_i^k C_j^l}{2^{k+l}} \int \cos^{i+j} 2x \,\mathrm{d}x.$$

If $\underline{i+j \text{ is odd}}$, we reduce the integral to case (b).
If $\underline{i+j \text{ is even}}$, we use half-angle identities again.

Example 2. Evaluate $\displaystyle\int \sin^4 x \,\mathrm{d}x$.

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以實例操作三角積分。

Solution.

Example 3 (page 482)**.** Compute the integrals $\displaystyle\int \tan x \,\mathrm{d}x$ and $\displaystyle\int \sec x \,\mathrm{d}x$.

重新複習 $\tan x$ 的積分; 另外也學習 $\sec x$ 的積分, 這裡介紹的技巧非常高超, 但式子很短, 一下就得到結果; 若你想要問是否有一個不像這裡介紹神來一筆的方法, 會在單元 7.4 介紹。

Solution.

☐ 上述方法太過技巧 (誰知道要乘什麼量), 但之後會學別的方式處理它 (想法比較自然)。

Example 4 (page 482)**.** Evaluate $\displaystyle\int \tan^m x \sec^n x \,\mathrm{d}x$, where $m, n \in \mathbb{N}$.

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第二類的三角積分要處理的是 $\tan x$ 與 $\sec x$ 各種次方的相乘之積分。這裡雖然用一般的記號直接計算其結果, 但是各位應該要理解其原理: 若 $\sec^n x$ 的次方是偶數, 或是 $\tan^m x$ 的次方是奇數, 那就用變數變換以及三角恆等式 $\sec^2 x = 1 + \tan^2 x$ 處理。若 $\sec^n x$ 的次方是奇數且 $\tan^m x$ 的次方是偶數, 那就要用分佈積分法處理。

Solution.

(a) If $\underline{n = 2k, k \in \mathbb{N}}$, then

$$\int \tan^m x \sec^n x \,\mathrm{d}x = \int \tan^m x \sec^{2k} x \,\mathrm{d}x = \underline{\qquad\qquad\qquad\qquad}$$

$$= \int \tan^m x (\tan^2 x + 1)^{k-1} \,\mathrm{d}\tan x = \int \tan^m x \sum_{i=0}^{k-1} C_i^{k-1} \tan^{2i} x \,\mathrm{d}\tan x$$

$$= \sum_{i=0}^{k-1} C_i^{k-1} \int \tan^{m+2i} x \,\mathrm{d}\tan x = \sum_{i=0}^{k-1} \frac{C_i^{k-1}}{m+2i+1} \tan^{m+2i+1} x + C.$$

(b) If $\underline{m = 2k+1, k \in \mathbb{N}}$, then

$$\int \tan^m x \sec^n x \,\mathrm{d}x = \int \tan^{2k+1} x \sec^n x \,\mathrm{d}x = \underline{\qquad\qquad\qquad\qquad}$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \,\mathrm{d}\sec x = \int \sum_{i=0}^{k} C_i^k \sec^{2(k-i)} (-1)^i \sec^{n-1} x \,\mathrm{d}\sec x$$

$$= \sum_{i=0}^{k} (-1)^i C_i^k \int \sec^{2(k-i)+n-1} x \,\mathrm{d}\sec x = \sum_{i=0}^{k} \frac{(-1)^i C_i^k}{2(k-i)+n} \sec^{2(k-i)+n} x + C.$$

(c) If $\underline{m = 2k, n = 2l+1}$, then

$$\begin{aligned}
I_m &= \int \tan^m x \sec^n x \,\mathrm{d}x = \int \tan^{2k} x \sec^{2l+1} x \,\mathrm{d}x = \int \tan^{2k-1} x \sec^{2l} x \,\mathrm{d}\sec x \\
&= \underline{\hspace{10cm}} \\
&= \tan^{2k-1} x \sec^{2l+1} x - \int \sec x (2k-1) \tan^{2k-2} \sec^2 x \sec^{2l} x \,\mathrm{d}x \\
&\quad - \int \sec x \tan^{2k-1} x (2l) \sec^{2l-1} x \sec x \tan x \,\mathrm{d}x \\
&= \tan^{2k-1} x \sec^{2l+1} x - (2k-1) \int \tan^{2k-2} \sec^{2l+3} x \,\mathrm{d}x - 2l \int \tan^{2k} x \sec^{2l+1} x \,\mathrm{d}x \\
&= \tan^{2k-1} x \sec^{2l+1} x - (2k-1) \int \tan^{2k-2} (\tan^2 x + 1) \sec^{2l+1} x \,\mathrm{d}x \\
&\quad - 2l \int \tan^{2k} x \sec^{2l+1} x \,\mathrm{d}x \\
&= \tan^{m-1} x \sec^n x - (m-1) I_m - (m-1) I_{m-2} - (n-1) I_m.
\end{aligned}$$

Hence we get

$$I_m = \frac{1}{m+n-1} \left(\tan^{m-1} x \sec^n x - (m-1) I_{m-2} \right),$$

and the reduction formula will reduce the integral to $\int \sec^n x \,\mathrm{d}x = \int \sec^{2l+1} x \,\mathrm{d}x$, $l \in \mathbb{N}$.

Exercise (page 481)**.** Evaluate $\displaystyle\int \tan^6 x \sec^4 x \,\mathrm{d}x$.

Exercise (page 482)**.** Evaluate $\displaystyle\int \tan^5 x \sec^7 x \,\mathrm{d}x$.

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這裡還要介紹另一種型式的三角積分，是 $\sin mx$ 與 $\cos nx$ 相乘的積分，只要用積化和差公式就能立刻得到結果。這裡得到的結果實際上和線性代數大有關係，它告知向量空間 $C([-\pi, \pi])$ 中會有一組單位正交基底，而這是傅利葉級數理論的開端。

Example 5 (page 484)**.** Evaluate the following integrals:

$$\int \sin mx \cos nx \,\mathrm{d}x; \quad \int \cos mx \cos nx \,\mathrm{d}x; \quad \int \sin mx \sin nx \,\mathrm{d}x.$$

Solution. Recall the following identities:

$$\begin{aligned}
\sin x \cos y &= \frac{1}{2}(\sin(x+y) + \sin(x-y)) \\
\cos x \cos y &= \frac{1}{2}(\cos(x+y) + \cos(x-y)) \\
\sin x \sin y &= -\frac{1}{2}(\cos(x+y) - \cos(x-y)).
\end{aligned}$$

If $\underline{m+n\neq 0}$ and $\underline{m-n\neq 0}$, then

$$\int \sin mx \cos nx \,\mathrm{d}x = \frac{1}{2}\int \sin((m+n)x) + \sin((m-n)x)\,\mathrm{d}x$$

$$= \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right.$$

$$\int \cos mx \cos nx \,\mathrm{d}x = \frac{1}{2}\int \cos((m+n)x) + \cos((m-n)x)\,\mathrm{d}x$$

$$= \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right.$$

$$\int \sin mx \sin nx \,\mathrm{d}x = -\frac{1}{2}\int \cos((m+n)x) - \cos((m-n)x)\,\mathrm{d}x$$

$$= \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right.$$

In particular,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin mx \cos nx \,\mathrm{d}x = \left\{ \begin{array}{l} \\ \\ \end{array} \right.$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos mx \cos nx \,\mathrm{d}x = \left\{ \begin{array}{l} \\ \\ \end{array} \right.$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin mx \sin nx \,\mathrm{d}x = \left\{ \begin{array}{l} \\ \\ \end{array} \right.$$

Hence $\{\sin mx, m \in \mathbb{N}$ and $\cos nx, n \in \mathbb{Z}, n \geq 0\}$ form an "orthonormal basis" in the function space $C([-\pi,\pi])$.

7.3 Trigonometric Substitution (page 486)

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Trigonometric identities are also useful to make substitutions for some radical functions.

根號內呈現變數平方與數字平方的加或減之積分都是屬於三角代換的範疇，只要把它和三角恆等式做完整地對應即可。

Table of Trigonometric Substitutions.

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta,\ 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

注意到三角代換 θ 的範圍，是爲了要「去掉絕對值」而設定；若選取別的範圍，去絕對值時必須補上負號，這樣做並不是不行，但是之後的計算會容易把自己困住。

Example 1 (page 487)**.** Find the area enclosed by the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Solution.

SddSCojKi-I

Example 2 (page 490)**.** Find $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\,\mathrm{d}x$.

Solution.

遇到根號內是二次三項式的時候，要和配方法聯想，配方法在代數上的目的是可以把一次項消掉。

Exercise (page 486)**.** Evaluate $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,\mathrm{d}x$.

Example 3 (page 488)**.** Find $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,\mathrm{d}x$.

Solution.

Exercise. Evaluate the integral $\displaystyle\int_1^2 \frac{1}{x^2\sqrt{1+x^2}}\,\mathrm{d}x$.

Example 4 (page 489)**.** Find $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x$, where $a > 0$.

ZBcB6VZLB3w

變數平方減常數平方，與 $\sec^2\theta - 1 = \tan^2\theta$ 聯想。

Solution.

Exercise. Find the integral $\displaystyle\int \frac{2}{x^3\sqrt{x^2-1}}\,\mathrm{d}x, \quad x > 1.$

Exercise. Evaluate the integral $\displaystyle\int \frac{x}{\sqrt{x^2+2x+2}}\,\mathrm{d}x$.

Example 5 (page 490)**.** Find $\displaystyle\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}\,\mathrm{d}x$.

Solution.

7.4 Integration of Rational Functions by Partial Fractions (page 493)

eMEE69427Sk

部份分式法是要處理有理函數的積分，它會有標準程序。這裡將介紹基本模型的積分該如何處理。這裡引用了最一般的記號，只是爲了論述的完整，看影片時，必須抓住的是處理積分的原則，不要被太多符號困住。

In this section we show how to integrate any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called *partial fractions* (部份分式).

Example 1. Discuss the integral $\int \frac{1}{(ax+b)^k}\,\mathrm{d}x$, where $k \in \mathbb{N}$.

Solution.

Example 2. Discuss the integral $\int \frac{Ax+B}{(ax^2+bx+c)^k}\,\mathrm{d}x$, where $b^2-4ac<0, k \in \mathbb{N}$.

Solution.

Integrate rational functions

Step 1: Perform the long division (長除法)

Definition 3 (page 494)**.** Consider a rational function $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

(a) If the degree of $P(x)$ is less than the degree of $Q(x)$, such a rational function $f(x)$ is called *proper*.

(b) If the degree of $P(x)$ is greater or equal to the degree of $Q(x)$, such a rational function $f(x)$ is called *improper*.

If $f(x)$ is *improper*, then we use the long division to get

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

and $\frac{R(x)}{Q(x)}$ is proper.

Step 2: Factor the denominator $Q(x)$ as a product of linear factors $(ax+b)$ and irreducible quadratic factors $ax^2+bx+c,\ b^2-4ac<0$. (因式分解)

Step 3: Express the proper rational function $\frac{R(x)}{Q(x)}$ as a sum of partial fractions. (拆成部份分式, 不同類型有不同的拆解法)

Definition 4 (page 494)**.** A rational function is called a *partial fraction* if it is of the form

$$\frac{A}{(ax+b)^n} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^n}.$$

(1) *$Q(x)$ is a product of distinct linear factors.* That is,

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k),$$

where *no* factor is repeated, then there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}.$$

For example, $\dfrac{2x^2-3x-1}{x(x+1)(x-1)} =$ ____________________, then

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部份分式法的第一步驟是長除法，若分子的次數大於分母，則長除法過後會有一部份是多項式，而多項式的積分就很容易處理。剩下的部份是真分式，之後的三個步驟都是在處理真分式的積分。

第二步驟是將分母因式分解，由代數的理論得知，多項式一定可以分解成一些一次式與一些二次三項式的乘積。

第三步驟就是部份分式法的重點，根據不同類型有不同的拆解方式。這裡必須好好體會。

最後一步驟就是按照前一個影片的方法逐一積分。

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根據因式的次方在做部份分式時，拆解的假設都不同。

(2) *$Q(x)$ is a product of linear factors, some of which are repeated.* Suppose the first linear factor (a_1x+b_1) is repeated r times, then instead of the single term $\frac{A_1}{a_1x+b_1}$, we would use

$$\frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_1x+b_1)^2}+\cdots+\frac{A_r}{(a_1x+b_1)^r}.$$

For example, $\dfrac{2x^2-x+3}{(x-1)^3}=$ ______________________________ .

(3) *$Q(x)$ contains irreducible quadratic factors, none of which is repeated.* That is, $Q(x)$ has the factor ax^2+bx+c, where $b^2-4ac<0$, then the expression for $\frac{R(x)}{Q(x)}$ will have a term of the form

$$\frac{Ax+B}{ax^2+bx+c},$$

and then we will use the formula

$$\int\frac{1}{x^2+a^2}\,\mathrm{d}x=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C.$$

(4) *$Q(x)$ contains a repeated irreducible quadratic factor.* If $Q(x)$ has the factor $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then instead of the single partial fraction, the sum

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

occurs in the $\frac{R(x)}{Q(x)}$.

For example, $\dfrac{x^3-x^2+2x+2}{(x^2+1)^2}=$ ______________________________ .

Example 5 (`Case` (1))**.** Show that $\int \sec\theta \,\mathrm{d}\theta = \ln|\sec\theta + \tan\theta| + C$.

Solution.

X_cw9yGJspA

我並沒有放錯例題，它的確是三角函數的積分，但它可用部份分式法得到結果。這裡的想法會比 7.2 介紹的方法容易理解，但是計算量稍大。

Example 6 (`Case` (2))**.** Find the integral $\int \frac{x^2+3x+2}{x^3-3x+2}\,\mathrm{d}x$.

Solution.

aSztwZ7Uyio

例題介紹第二類型的部份分式該如何處理。

Example 7 (`Case` (3))**.** Evaluate the integral $\int \frac{2x^2+5x+3}{(x^2+2x+2)(x-1)}\,\mathrm{d}x$.

Solution.

_EPjrXUpVa8

例題介紹第三類型的部份分式該如何處理。

3QtcrPI0zZk

Example 8. Find the integral $\int \frac{2x+1}{(x^2+1)^2}\,\mathrm{d}x$.

Solution.

例題介紹第四類型的部份分式該如何處理。

Rationalizing substitutions, page 500

Some nonrational functions can be changed into rational functions by means of appropriate substitutions. In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, then the substitution $u = \sqrt[n]{g(x)}$ may be effective.

3_cTz440zNM

Example 9 (page 500)**.** Evaluate $\int \frac{\sqrt{x+4}}{x}\,\mathrm{d}x$.

Solution.

當積分的函數內部有根號，而且內部不能用三角替換法處理的時候，可以考慮直接把整個根號令成變數 u，有時候它可以轉變成對 u 而言的有理函數，那就再用部份分式法處理。

Convert rational functions of $\sin x$ and $\cos x$, page 502

The German mathematician Karl Weierstrass (1815-1897) noticed that the substitution $t = \tan(\frac{x}{2})$ will convert any rational function of $\sin x$ and $\cos x$ into an ordinary rational function of t.

0AOC8TAz7HI

所有以 $\sin x$ 與 $\cos x$ 爲變數的有理函數積分，都可以透過「萬能公式」處理，只要令 $t = \tan(\frac{x}{2})$，就可以把積分變換成以 t 爲變數的有理函數。雖然這招叫做萬能公式，但是處理過程非常繁瑣，所以不建議直接使用，在其他的積分方法都想不出來的情況下，不得已再使用這個大絕。

If $t = \tan(\frac{x}{2}), -\pi < x < \pi$, then we have

$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}} \quad \text{and} \quad \sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}.$$

By double-angle formula, we get

$$\cos x = \frac{1-t^2}{1+t^2} \quad \text{and} \quad \sin x = \frac{2t}{1+t^2}.$$

Furthermore, we can compute

$$\mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t.$$

☐ 被積函數是 $\sin x$ 與 $\cos x$ 組成的有理函數, 可透過變數代換 $t = \tan(\frac{x}{2})$ 處理。

Example 10. Find the integral $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{2+\cos x}\,\mathrm{d}x$.

Solution.

7.5 Strategy for Integration (page 503)

DqJHt2gnHsc

這一節回顧到目前爲止所學到的五種積分技巧，必須熟悉。

關於這裡所列的積分表，我並沒有背它，我都是在遇到問題時，直接透過變數變換的原則很快速地改寫就能得到結果。

有的時候一個函數的積分處理方法有很多種，所以平時應多看多試，培養起你對積分的各種直覺等。

We have learned the following techniques to integrate a function:

- Substitution rule, section 5.5.
- Integration parts, section 7.1.
- Trigonometric Integrals 7.2.
- Trigonometric substitution, section 7.3.
- Partial fractions, section 7.4.

In this section, we present a collection of miscellaneous integrals in random and the main challenge is to recognize which technique or formula to use.

Table of Indefinite Integrals (page 503)**.**

$$\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}\left(\frac{x}{a}\right) + C, \quad a>0$$

$$\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C \qquad \int \frac{1}{\sqrt{x^2\pm a^2}}\,\mathrm{d}x = \ln\left|x+\sqrt{x^2\pm a^2}\right| + C$$

Once you are armed with these basic integration formulae, if you don't immediately see how to attack a given integral, you might try the following four-steps strategy.

(1) Simplify the integrand if possible. Use algebraic manipulation or trigonometric identities to simplify the integrand.

$$\int \sqrt{x}(1+\sqrt{x})\,\mathrm{d}x =$$

$$\int \frac{\tan\theta}{\sec^2\theta}\,\mathrm{d}\theta =$$

$$\int (\sin\theta+\cos\theta)^2\,\mathrm{d}\theta =$$

jIjV4J7AOHM

有時候遇到有理函數的積分，不要急著用部份分式法，雖然它有標準流程，但是過程較爲繁瑣。當分子與分母之間的關聯互爲微分與積分的時候，那就可以直接使用變數變換法。而分母是變數平方減常數平方的形式，也可以嘗試用三角代換處理。

(2) Look for an obvious substitution.

$$\int \frac{x}{x^2-1}\,\mathrm{d}x =$$

(3) Classify the integrand according to its form.

(a) Trigonometric function: product of powers of $\sin x$ and $\cos x$, of $\tan x$ and $\sec x$, or $\cot x$ and $\csc x$.

(b) Rational functions: $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

(c) Integration by parts: product of a power of a polynomial and a transcendental function (trigonometric, exponential, or logarithmic).

(d) Radicals: $\sqrt{\pm x^2 \pm a^2}$, $\sqrt[n]{ax+b}$, or $\sqrt[n]{g(x)}$.

(4) Try again: remember that there are basically only two methods of integration: substitution and parts.

(a) Try substitution: inspiration, ingenuity, desperation.

(b) Try parts: it is sometimes effective on single function, such as $\sin^{-1} x$, $\tan^{-1} x$, $\ln x$ (inverse functions).

(c) Manipulate the integrand:

$$\int \frac{1}{1-\cos x}\,\mathrm{d}x =$$

$$=$$

(d) Relate the problem to previous problems.

$$\int \tan^2 x \sec x \,\mathrm{d}x =$$

(e) Use several methods: substitution, integration by parts, etc.

Example 1 (page 505)**.** Compute $\int \frac{\tan^3 x}{\cos^3 x}\,\mathrm{d}x$.

I3T1-01JeqI

積分技巧千變萬化，微積分課程中只學了其中五種技巧，而這些技巧可以涵蓋的類型就非常多了。所以各位現在開始應該盡量著手思考如何處理每個積分。

Solution.

Solution 2.

Example 2 (page 506)**.** Compute $\int \mathrm{e}^{\sqrt{x}}\,\mathrm{d}x$.

Solution.

Example 3 (page 506)**.** Compute $\int \sqrt{\dfrac{1-x}{1+x}}\,\mathrm{d}x$.

Solution.

Can we integrate all continuous functions?

5MMnk1hcWxA

想清楚不可積、不會積與積不出來的差別。不可積分是定積分之黎曼和極限不存在；不會積指的是個人的積分能力不足；積不出來的意思是反導函數無法表示成基本函數的型式。這裡列出幾種積不出來的函數。

對於積不出來的函數，有時我們還是可以問它特定的積分值，因爲它積不出來，所以不可能把反導函數明確表示出，這時就要透過其他的數學理論去處理。

Definition 4 (page 506)**.** *Elementary functions* are all polynomials, rational functions, power functions, exponential functions, logarithmic functions, trigonometric and inverse trigonometric functions, hyperbolic and inverse hyperbolic functions, and all functions that can be obtained from these by the five operations of addition, substraction, multiplication, division, and composition.

If $f(x)$ is an elementary function, then $f'(x)$ is an elementary function, but $\int f(x)\,\mathrm{d}x$ need not be an elementary function. For example,

(1) $\displaystyle\int \sqrt{1-2\sin^2 x}\,\mathrm{d}x$: elliptic integral (橢圓積分), 它是計算橢圓弧長的積分表達式 (8.1 會介紹如何計算曲線的長度)。

(2) $\displaystyle\frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t$: error function (誤差函數), 常用於機率統計 (常態分布) 與工程。

(3) $\displaystyle\int \sin(x^2)\,\mathrm{d}x, \int \cos(x^2)\,\mathrm{d}x$: Fresnel integral (菲涅耳積分), 與誤差函數有關聯。

(4) $\displaystyle\int \frac{\sin x}{x}\,\mathrm{d}x, \int \cos(\mathrm{e}^x)\,\mathrm{d}x$: sine integral function (cosine integral function)。

(5) $\displaystyle\int \frac{\mathrm{e}^x}{x}\,\mathrm{d}x$: exponential integral (指數積分)。

(6) $\displaystyle\int \frac{1}{\ln x}\,\mathrm{d}x$: logarithmic integral (對數積分)。

(7) $\displaystyle\int \sqrt{x^3+1}\,\mathrm{d}x$: 此積分可以化簡成和橢圓積分有關。

雖然上述函數的積分無法表示成基本函數的型式, 但是有時代入特殊的上、下限可以透過其他分析方式 (多變數微積分、複變函數論、微分方程) 等求得明確的數值。例如:

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}, \quad \int_{-\infty}^{\infty} \sin(x^2)\,\mathrm{d}x = \sqrt{\frac{\pi}{2}}, \quad \int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \pi.$$

7.8 Improper Integrals (page 527)

In this section we extend the concept of a definite integral to the case where the interval is infinite and also to the case where f has an infinite discontinuity in $[a, b]$. In either case the integral is called an *improper integral* (瑕積分).

nYQsfBC-B_I

這一節要處理瑕積分。瑕積分有兩類，首先要觀察的是積分範圍是無窮的情況，因爲在任何有限的範圍內都可以求定積分，所以這類瑕積分的定義就是對範圍再取一次極限。

Type 1: Infinite Intervals

Definition of an Improper Integral of Type 1 (page 528)**.**

(a) If $\int_a^t f(x)\,\mathrm{d}x$ exists for every number $t \geq a$, then $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x$ provided this limits exists (as a finite number).

(b) If $\int_t^b f(x)\,\mathrm{d}x$ exists for every number $t \leq b$, then $\displaystyle\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{t\to-\infty}\int_t^b f(x)\,\mathrm{d}x$ provided this limits exists (as a finite number).

若積分範圍是整個實數軸 $\mathbb{R}$，則將它分成兩半形成兩個瑕積分，當確定各別的瑕積分都存在時，才規定定義在 $\mathbb{R}$ 上的函數瑕積分收斂；否則稱爲發散。

The improper integrals $\int_a^\infty f(x)\,\mathrm{d}x$ and $\int_{-\infty}^b f(x)\,\mathrm{d}x$ are called *convergent* (收斂) if the corresponding limits exists and *divergent* (發散) if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,\mathrm{d}x$ and $\int_{-\infty}^a f(x)\,\mathrm{d}x$ convergent, then we define

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^a f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x$$

In part (c) any real number a can be used.

Example 1 (page 530)**.** Discuss the areas of the infinite region $\mathcal{R}$ under the curve $y = \frac{1}{x^p}, p > 0$ and to the right $x = 1$.

vTUfUp2Cz-8

這個基本模型必須要徹底理解。若用面積的角度理解它的話， p 的次數要大，區域面積才可能有限。而瑕積分好壞的臨界點是 $p = 1$。

Solution.

Type 2: Discontinuous Integrands

swYJgvfbkoA

第二類瑕積分要處理的是函數在某一點衝到無限大的情況。 這類瑕積分,定義方式是在瑕點的附近選擇一點,那麼定積分就可以處理,然後再追問這個點趨近於瑕點的時候,定積分的極限是否存在。

這裡先定義單邊的瑕積分,而對於不連續點的函數之瑕積分,必須確定左極限與右極限瑕積分都必須存在下,才規定整體的瑕積分收斂,否則稱爲發散。

對於第二類瑕積分,也是必須想清楚這個基本模型,若用面積的角度理解它的話,p 的次數要小,區域面積才可能有限。而瑕積分好壞的臨界點是 $p=1$。

Definition of an Improper Integral of Type 2 (page 531)**.**

(a) If f is continuous on $[a,b)$ and is discontinuous at b, then $\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to b^-}\int_a^t f(x)\,\mathrm{d}x$ if this limits exists (as a finite number).

(b) If f is continuous on $(a,b]$ and is discontinuous at a, then $\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to a^+}\int_t^b f(x)\,\mathrm{d}x$ if this limits exists (as a finite number).

The improper integrals $\int_a^b f(x)\,\mathrm{d}x$ is called *convergent* (收斂) if the corresponding limits exists and *divergent* (發散) if the limit does not exist.

(c) If f has a discontinuity at c, where $a<c<b$, and both $\int_a^c f(x)\,\mathrm{d}x$ and $\int_c^b f(x)\,\mathrm{d}x$ convergent, then we define $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$.

Example 2 (page 535)**.** Discuss the areas of the region $\mathcal{R}$ under the curve $y=\frac{1}{x^p}, p>0$, and between $x=0$ and $x=1$.

Solution.

Rvet_Spm_r4

多數同學在學習這個單元的時候,會被怎麼一下 $p>1$ 一下又 $p<1$ 困住,實際上這兩類的瑕積分與標準模型彼此是互相等價的。只要透過圖形的理解,就能清楚地知道兩者的關係,這樣就不會被瑕積分的收斂條件困惑住。

Example 3. Compare Example 1 with Example 2.

Solution.

A Comparison Test for Improper Integrals

Comparison Theorem (page 533)**.** *Suppose that f and g are continuous functions with $f(x) \ge g(x) \ge 0$ for $x \ge a$.*

(a) *If $\int_a^\infty f(x)\,\mathrm{d}x$ is convergent, then $\int_a^\infty g(x)\,\mathrm{d}x$ is convergent.*

(b) *If $\int_a^\infty g(x)\,\mathrm{d}x$ is divergent, then $\int_a^\infty f(x)\,\mathrm{d}x$ is divergent.*

7wJB_9smHU4

對於複雜函數的瑕積分，我們在意的是它是否收斂或發散，而不是追問其明確的積分值。判定瑕積分的好壞，可以用比較判別法處理：設法找到一個較爲容易的函數，兩者有大小關係，而且簡單的函數的瑕積分可以知道收斂或發散，那就可以得到複雜函數的瑕積分收斂或發散。

☐ 定理的條件「$f(x) \ge g(x) \ge 0$」, 函數「非負」是必要的。

☐ 定理敘述中「for $x \ge a$」可以改成「for some $x \ge b$, $b \ge a$」。

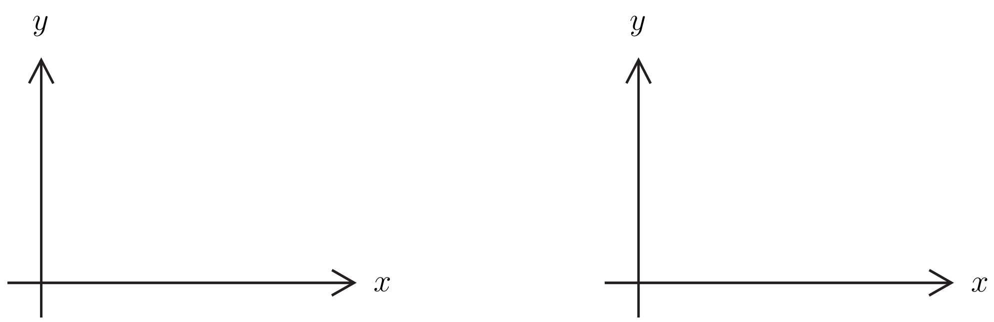

Figure 1: Comparison Theorem.

Example 4.

(a) Find the values of α for which the improper integral $\displaystyle\int_1^\infty \frac{1}{x^\alpha(1+\sqrt{x})}\,\mathrm{d}x$ converges.

(b) Evaluate the integral $\displaystyle\int_0^\pi \sec^2 x\,\mathrm{d}x$.

Solution.

cPjncIHOUz8

Example 5 (page 535)**.** Let $I_n = \int_0^\infty x^n e^{-x}\,dx$. Find the reduction formula.

Solution.

這個函數與伽瑪函數相關 (Gamma function)， 它是一個把階乘連續化的過程。

MoU_SKXztCQ

Example 6 (page 535)**.**

(a) Determine the values of $\alpha > 0$ such that $\int_1^\infty \frac{\ln x}{x^\alpha}\,dx$ is convergent.

(b) Find the integral $\int_1^\infty \frac{\ln x}{x^3}\,dx$.

Solution.

當 x 很大的時候對數函數比任何冪次函數都還要跑得慢; 所以跟冪次函數借一點點去吸收對數，就可以得知其收斂的條件。

PtYMevi1y30

Example 7. Evaluate the improper integral $\int_0^2 \frac{\sqrt{x(2-x)}}{x}\,dx$.

Solution.

瑕積分的問題，首先要確定瑕點在哪裡，是無窮遠或是某個特別的點，再針對它是第一類或第二類瑕積分處理它是收斂或發散。這個例題是可以確實透過積分的方式處理瑕積分的值。