Chapter 6 Applications of Integration

6.1 Areas Between Curves, page 422

Theorem 1 (page 422)**.** *The area A of the region bounded by the curves $y = f(x), y = g(x)$, and the lines $x = a, x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all $x \in [a, b]$, is*

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))\Delta x = \int_a^b (f(x) - g(x))\,\mathrm{d}x.$$

Proof. This is because

$$\begin{aligned} A &= (\text{area under } y = f(x)) - (\text{area under } y = g(x)) \\ &= \int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x = \int_a^b (f(x) - g(x))\,\mathrm{d}x. \end{aligned}$$

□

QNYoSbMT1Dw

注意到定積分代表的是有方向與有符號的面積，所以若要呈現面積大小(正的量)，則函數必須大減小，而且積分是由左至右。

所以給定兩函數，必須了解大小關係，通常是把交點（兩者相等的地方）找出來，再分析交點之間的大小，所以必須要會一些解方程式的能力。

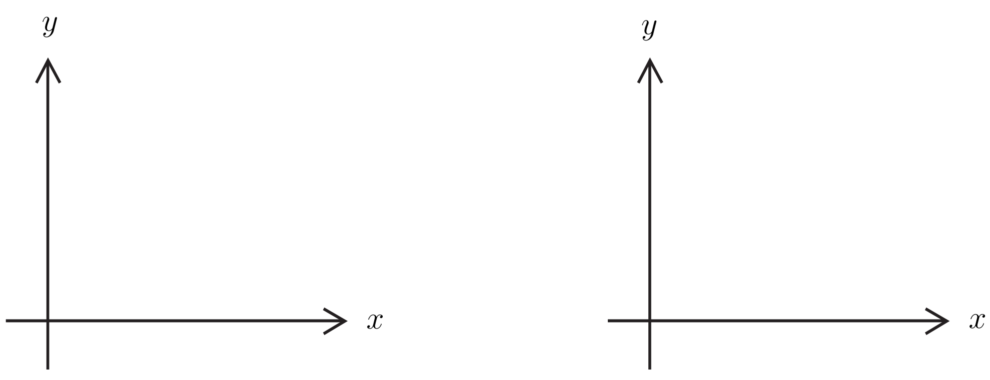
Theorem 2 (page 425)**.** *The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is*

$$A = \int_a^b |f(x) - g(x)|\,\mathrm{d}x.$$

Proof. This is because

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{if } f(x) \geq g(x) \\ g(x) - f(x) & \text{if } g(x) \geq f(x). \end{cases}$$

□



Area =

Area =

Figure 1: The area formula.

h12ij4oQLH8

計算面積時，必須判斷函數在範圍內誰大誰小，再確實計算面積。

Example 3 (page 427)**.** Sketch the region enclosed by $y = \tan x, y = 2\sin x, -\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$ and find its area.

Solution.

KCREbeIUBwM

有的時候區域的邊界對 y 而言表示成函數，那麼計算區域面積時可以考慮「積 y」; 這個例子如果要對 x 積分的話，必須分段處理。

Example 4 (page 426)**.** Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution.

6.2 Volumes, page 438

Definition 1 (page 439)**.** Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the *volume* (體積) of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b A(x)\,\mathrm{d}x.$$

8dF5_4KkEBk

這裡介紹圓盤法處理旋轉體體積。列出旋轉體體積公式的時候，必須想清楚截面的形狀是圓盤或是環狀，再根據形狀確實列式。

Definition 2 (page 443)**.** The solids are obtained by revolving a region about a line is called *solids of revolution* (實心旋轉體).

In general, we calculate the volume of a solid of revolution by the formula

$$V = \int_a^b A(x)\,\mathrm{d}x \quad \text{or} \quad V = \int_c^d A(y)\,\mathrm{d}y,$$

where

- If the cross-section is a disk, then $A = \pi(\text{radius})^2$.
- If the cross-section is a washer, then $A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$.

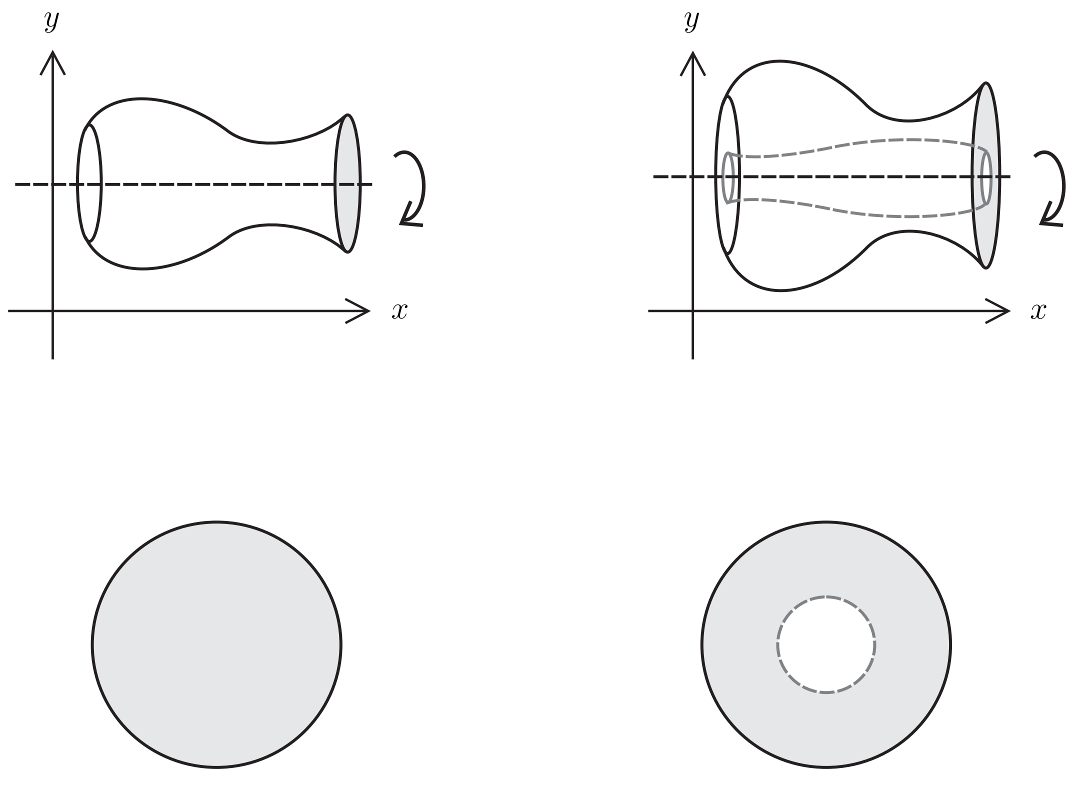


Figure 1: The volume formula of solids of revolution.

QGzg7fM09xI

利用積分的方式確實驗證以前背過的實心球體體積。

Example 3 (page 439)**.** Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Solution.

實心甜甜圈的體積也可以用物理的觀點去了解它：截面面積乘上截面的質心繞軸旋轉後的總路徑。

Example 4 (page 448)**.** Compute the volume of the solid torus.

Solution.

VHlbIg60ex8

注意旋轉軸所在位置的不同，則看待問題的方法還有計算截面面積的方式都必須重新確立。

Example 5 (page 442)**.** Consider the region $\mathcal{R}$ enclosed by the curves $y = x$ and $y = x^2$.

(a) Find the volume of the solid obtained by rotating the region about the line $y = 2$.

(b) Find the volume of the solid obtained by rotating the region about the line $x = -1$.

Solution.

We now find the volumes of two solids that are *not* solids of revolution.

Example 6 (page 445)**.** Find the volume of a pyramid whose base is a square with side L and whose height is h.

17tiKoRlHRk

以下幾個實心物體並非旋轉體，討論體積時，必須把相似形的關係找出來。

Solution.

Example 7 (page 446)**.** A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.

從影片中應學習到楔形物的截面形狀以及如何建立兩截面之間相似形的關係。

Solution.

Example 8 (page 449)**.** Find the volume common to two circular cylinders, each with radius r, if the axis of the cylinder intersect at right angles.

-5xK-I382ms

兩個圓柱互相垂直地交集出的實心物體可能不是那麼好想像，課堂中會呈現相關道具，在課堂中確實理解其形狀。

另一方面，想清楚這個幾何物件與金字塔形狀的差異。

Solution.

TqspHOVq4LA

這裡雖然是對旋轉體的各種可能統整列出公式，但各位不應死記公式，而是確實畫圖理解截面的形狀，然後把相關的量寫出，就會覺得很自然。

The volume formula of solid of revolution

(a) Region under $f(x) > 0$; rotate about x-axis.

(b) Region between $f(x)$ and $g(x)$, $f(x) > g(x) > 0$; rotate about x-axis.

(c) Region under $f(x) > 0$; rotate about the line $y = c$.

(d) Region between $f(x)$ and $g(x)$, $f(x) > g(x) > c$; rotate about the line $y = c$.

6.3 Volumes by Cylindrical Shells, page 449

Definition 1 (page 437)**.** The volume of the solid obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)\Delta x = \int_a^b 2\pi x f(x)\,\mathrm{d}x, \quad \text{where} \quad 0 \le a < b.$$

This method is called *cylindrical shells method* (柱殼法).

5xvlqhL1m5Y

柱殼法是另一種計算體積的方法， 把幾何形體想成是以旋轉軸爲中心軸的各種圓柱疊加而成。各位在學這部份的時候應想清楚柱殼法與圓盤法的差別，特別是旋轉軸的位置。

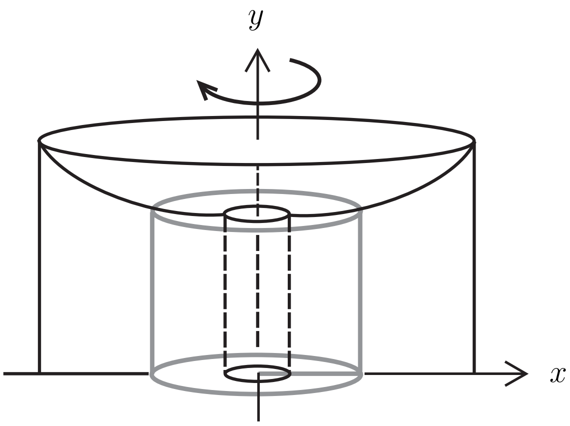


Figure 1: The volume formula by cylindrical shells (rotate about y-axis).

Example 2 (page 453)**.** Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = \sin(x^2)$ and $y = 0$ for $0 \le x \le \sqrt{\pi}$.

Solution.

1KRXgLf5BmM

Example 3. Find the volume of the solid obtained by rotating about $x = -1$ the region bounded by $y = 6x^2, x = 1$, and $y = 0$.

目前已經學到計算旋轉體體積的方法有圓盤法與柱殼法, 影片中確實利用兩種方法計算得到相同的結果, 各位應從中看清楚兩種方法的異同與優劣性。

這裡也要必須注意旋轉軸的位置。

Solution.

☐ 計算旋轉體體積問題, 除了區域要確定以外, 對哪一個軸旋轉也很重要。

☐ 目前無法解決太多用柱殼法所列出的體積問題, 這是因爲尚有一些積分技巧未學。

The volume formula of solid of revolution

柱殼法的統整。實際上區域可以更任意, 例如介在 $g(x)$ 與 $f(x)$ 之間的區域。

(a) Region under $f(x) > 0, x \in [a, b]$; rotate about y-axis.

(b) Region under $f(x) > 0, x \in [a, b]$; rotate about $x = c, c < a$.

6.5 Average Value of a Function (page 461)

Definition 1 (page 461)**.** We define the *average value of f* (平均值) on the interval $[a, b]$ as

$$f_{\text{ave}} = \lim_{n\to\infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x.$$

7uhgMX2XJOA

中學以前的是有限個數的平均值，現在將其概念過渡到定義函數的平均。

Example 2. Find the average of $f(x) = \sin x$ on $[0, \pi]$.

Solution.

例題看似平凡無奇，但是它和著名的布豐投針 (Buffon's Needle) 相關，有興趣者可以繼續網路搜尋相關資訊。

The Mean Value Theorem for Integrals (pgae 462)**.** *If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that*

$$f(c) = f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x. \qquad \left(or \int_a^b f(x)\,\mathrm{d}x = f(c)(b-a). \right)$$

閉區間上的連續函數有積分版本的平均值定理，這個結果其實是非常自然的，它告知區域面積會和以 $[a, b]$ 爲底、某個函數值爲高的矩形面積一樣。

Proof. Consider $F(x) = \int_a^x f(t)\,\mathrm{d}t$. Since $f(x)$ is continuous on $[a, b]$, $F(x)$ is continuous on $[a, b]$ and differentiable on (a, b). By the Mean Value Theorem, there exists $c \in (a, b)$ such that ____________________. By the Fundamental Theorem of Calculus, we have ____________. Hence

$$f(c) =$$

□

Example 3. Suppose that $f(x)$ is an increasing and continuous function on $[a, b]$. Find the line $y = L$ such that $\int_a^b |f(x) - L|\,\mathrm{d}x$ is minimum.

e6puM_wfof0

以前在敘述統計時所學的中位數，現在賦予它另一個深刻的意義。此外，分析這個問題的方法也值得一學，像是被積分函數如何拆絕對值、微積分基本定理的使用，最佳化問題的處理、函數與反函數之間的關係，把之前學到的很多概念串連起來。而影片 13:25 之後如何建立直觀的看法也值得一學。

Solution.

NAH6_A7Ietk

Example 4. Suppose that $f(x)$ is a continuous function $f(x)$ on $[a, b]$. Find the line $y = L$ such that $\int_a^b (f(x) - L)^2 \, dx$ is minimum.

同樣地，平均數也可以賦予另一個深刻的「變分」的意義; 在影片 6:15 之後，藉由這個例子將平均數與變異數 (variance) 這兩個統計量再做更緊密的連繫。

Solution.

☐ 中位數 (median) 與平均數 (average) 的意義。