Chapter 5 Integrals

5.1 Areas and Distances, page 360

The Area Problem, page 360

Example 1. Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.

Solution.

EoF_wAlXL7E

關於拋物線下區域面積，早在阿基米德 (公元前 287–212) 的時候就知道了，那時阿基米德雖然沒有像我們一樣已經非常清楚極限的理論，但是不斷地分割再求和以估計並逼近區域面積這個概念就已成形。

這裡介紹的方法利用了平方和的公式、極限的四則運算還有夾擠定理，可以將拋物線下的面積解釋得一清二楚。

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Definition 2 (page 365)**.** The *area* (面積) A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} (f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x).$$

由前一個例子的討論，引出連續函數的圖形與 x-軸所圍的區域面積，可以利用分割、樣本點、取和、求極限這四個步驟定義；下一節將仔細討論相關理論。

It can also be shown that we get the same value if we use left endpoints:

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} (f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x).$$

In fact, instead of using left endpoints or right endpoints, we could take the height of the i-th rectangle to be the value of f at *any* number x_i^* in the i-th subinterval $[x_{i-1}, x_i]$. We call numbers $x_1^*, x_2^*, \ldots, x_n^*$ the *sample points* (樣本點).

In general, we form *lower sums* (下和) (and *upper sums*, 上和) by choosing the sample points x_i^* so that $f(x_i^*)$ is the minimum (and maximum) value of f on the i-th subinterval.

The Distance Problem, page 367

在物理上，物體的速度與位移之間的關係也將引出下一節定積分的概念。

We can find the distance traveled by an object during a certain time period if the velocity of the object is known at all times.

5.2 The Definite Integral, page 378

Definition of a Definite Integral (page 378)**.** If f is a function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n-subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0 = a, x_1, x_2, \ldots, x_n = b$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any *sample points* (樣本點) in these subintervals, so x_i^* lies in the i-th subinterval $[x_{i-1}, x_i]$. Then the *definite integral* (定積分) *of f from a to b* is

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is *integrable* (可積分的) on $[a, b]$.

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函數與 x-軸所圍的區域面積，利用分割樣本點取和求極限四個步驟定義。這裡應強調的是，若函數是可積分的，必須要求樣本點在做任意的選取之下，極限都必須存在並且相等，這算是非常高的要求。

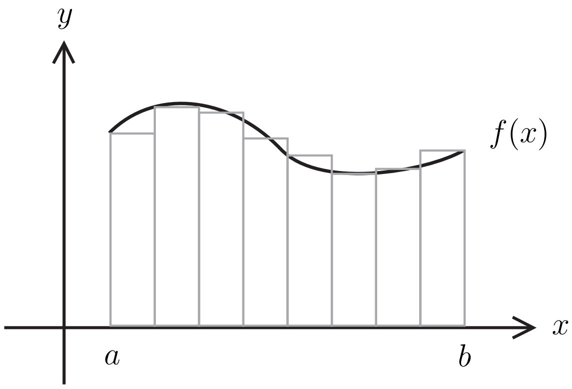

Figure 1: Definition of a definite integral.

The precise meaning of the limit that defines the integral is as follows:

定積分的概念是極限，所以可以用極限的精確定義重新描述。

> For every number $\varepsilon > 0$, there is an integer N such that
>
> $$\left| \int_a^b f(x)\,\mathrm{d}x - \sum_{i=1}^{n} f(x_i^*)\Delta x \right| < \varepsilon$$
>
> for every integer $n > N$ and for every choice of x_i^* in $[x_{i-1}, x_i]$.

There are some notations we should know:

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確實認識積分符號。

$$\int_a^b f(x)\,\mathrm{d}x$$

integral sign:
integrand:
limits of integration:
lower limit (下限):
upper limit (上限):

☐ The procedure of calculating an integral is called *integration*.

☐ The $\mathrm{d}x$ simply indicates that the independent variable is x. (dummy variable)

- ☐ The definite integral $\int_a^b f(x)\,\mathrm{d}x$ is a number; it does not depend on x.
- ☐ The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a *Riemann sum* (黎曼和).
- ☐ The geometric meaning of $\int_a^b f(x)\,\mathrm{d}x$ is the *net area* of $y = f(x)$ from a to b.
- ☐ In fact, the subinterval widths are not necessary equal width.

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

- ☐ Not all functions are integrable. For example, the Dirichlet function, or $f(x) = \frac{1}{x}$ on $0 < x \le 1$.

vcgx3IUxAdE

Example 1. Show that $f(x) = \frac{1}{x}$ on $0 < x \le 1$ is not integrable.

Solution.

在這個例子中，若要證明定積分不存在，只要證明在分割之下，找到一組樣本點使其黎曼和取極限後不存在即可。所以觀察的重點是如何論述黎曼和取極限不存在。

Theorem 2 (page 380)**.** *If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuous, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,\mathrm{d}x$ exists.*

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If f is integrable on $[a, b]$, then the limit of Riemann sum exists and gives the same value no matter how we choose the sample points x_i^*. To simplify the calculation of the integral, we often taken the sample points to be right endpoints. Then $x_i^* = x_i$ and the definition of an integral simplifies as follows.

閉區間上的連續函數可積分，這件事將在高等微積分課程中學習如何嚴格地證明。而定積分存在的函數可以稍微爛一點，若函數只有有限個跳躍不連續點，定積分仍存在。

Theorem 3 (page 380)**.** *If f is integrable on $[a, b]$, then*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Example 4 (page 383)**.** Set up an expression for $\int_1^3 \mathrm{e}^x \, \mathrm{d}x$ as a limit of sums and evaluate it.

Solution.

因爲函數是指數函數，所以可以用等比級數公式將黎曼和簡單地表達。最後處理極限值也是一門學問，因爲 n 是正整數，把問題過渡成以 x (實數) 爲變數，再用羅必達法則處理。

從這個例子會發現：用定義計算定積分是件不簡單的事，可想而知的是，如果函數再複雜一點，很難會有什麼公式把黎曼和簡單地表達以算出極限。所以在之後的章節，我們將介紹別的方法快速得到定積分，以取代這種土法煉鋼式的用定義計算積分值。

XeoEdy3q6Tg

Example 5. Change the following limits of sums as integrals:

$$\lim_{n\to\infty} n^2 \left(\frac{1}{n^3} + \frac{1}{(n+1)^3} + \cdots + \frac{1}{(n+(2n-1))^3} \right).$$

Solution.

有些極限問題可以把它轉變成定積分的型式，然後等到之後的章節學完就可以快速求得積分值。至於如何將極限轉換成定積分，可以先想辦法造出 $\Delta x = \frac{1}{n}$ 這個量，再去設定函數、樣本點、上下限得到定積分。要注意的是，最後寫出來的定積分表示法不唯一。

☐ **Example 1** in section 5.1 is also an evaluating integral of $\int_0^1 x^2 \, \mathrm{d}x$.

Properties of the Definite Integral

M_ybjcBk_9w

Properties of the Integral (page 385–387).

試著把積分的性質與「有向面積」或是「帶有正負號的面積」對應。而產生正負號的可能性有兩種，一個是函數的正負，另一個是積分的上下限互換。

性質 (2) 與 (3) 與線性代數有關：積分是一種線性變換。

(1) $\int_a^b c\,\mathrm{d}x = c(b-a)$, *where c is any constant.*

(2) $\int_a^b (f(x) \pm g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$.

(3) $\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$, *where c is any constant.*

(4) $\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x$.

(5) *If $f(x) \geq 0$ for $a \leq x \leq b$, then* $\int_a^b f(x)\,\mathrm{d}x \geq 0$.

(6) *If $f(x) \geq g(x)$ for $a \leq x \leq b$, then* $\int_a^b f(x)\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x$.

(7) *If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then* $m(b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M(b-a)$.

除了學習如何搭配上面的積分性質完成論述之外，試著將這個例子和「三角不等式」做聯想，就會覺得這個不等式很自然。

Example 6 (page 391)**.** If f is continuous on $[a,b]$, show that

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x.$$

Solution.

5.3 The Fundamental Theorem of Calculus, page 392

The Fundamental Theorem of Calculus, Part 1 (page 394). *If f is continuous on $[a, b]$, then the function g defined by*

$$g(x) = \int_a^x f(t)\,\mathrm{d}t \quad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

eamA03UJ_kQ

注意到定理敘述中 $g(x)$ 這個函數的變數 x 對應到積分的上限, 因爲 x 符號被借用了, 所以積分的啞吧變數改用別的符號 t 表示。

看數學證明的時候, 可以把每一個式子搭配圖形的意義就變得容易理解。

Proof. For x and $x + h$ in (a, b), we have

$$g(x+h) - g(x) =$$

$$=$$

so for $h \neq 0$,

$$\frac{g(x+h) - g(x)}{h} =$$

Assume that $h > 0$. Since f is continuous on $[x, x + h]$, the ____________________ says that there are $u, v \in [x, x + h]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x, x + h]$. So

$$mh \le \int_x^{x+h} f(t)\,\mathrm{d}t \le Mh \Rightarrow$$

Now we let $h \to 0$, then $u \to x$ and $v \to x$, so

$$\lim_{h\to 0} f(u) = f(\lim_{u\to x} u) = f(x) \quad \text{and} \quad \lim_{h\to 0} f(v) = f(\lim_{v\to x} v) = f(x)$$

because f is continuous at x. By the ______________, we have

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f(x).$$

If $x = a$ or b, the above discussion can be modified by considering one-sided limit. Since g is differentiable on $[a, b]$, g is continuous on $[a, b]$. □

Remark 1. The Fundamental Theorem of Calculus, Part 1, can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x)$$

when f is continuous.

微積分基本定理第一部份想傳達的概念是: 微分和積分是互逆的運算, 把連續函數先積分再微分就會還原。

cAwFW1E7_D0

Example 2 (page 395)**.** Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\,\mathrm{d}t$.

Solution.

這個例子觀察的重點是: 當積分上限不是 x 的時候, 而 $h'(x)$ 是問 $h(x)$ 對 x 的變化率, 所以會有鏈鎖律的關係: 第一層的微分是用微積分基本定理, 第二層的微分是積分上限對 x 的變化, 兩者相乘。

Example 3. Find the derivative of $h(x) = \int_0^{\sin x} \sqrt{1+r^3}\,\mathrm{d}r$.

Solution.

Y8eCa57yi9E

微積分基本定理第二部份是說: 如果有辦法找到連續函數 $f(x)$ 的隨便一個反導函數 $F(x)$, 那麼函數 $f(x)$ 在 $[a, b]$ 之間的定積分值和反導函數代入上限與下限後的值相減結果一樣。這個定理的建立, 就可取代前一節所述利用定義計算定積分帶來的各種不便性。下一個單元我們就要開始追問每一個函數的反導函數是什麼? 又該如何尋找?

The Fundamental Theorem of Calculus, Part 2 (page 396)**.** *If f is continuous on $[a, b]$, then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a),$$

where $F(x)$ is any antiderivative of $f(x)$, that is, a function such that $F'(x) = f(x)$.

Proof. Let $g(x) = \int_a^x f(t)\,\mathrm{d}t$. From the Fundamental Theorem of Calculus, Part 1, we know $g'(x) = f(x)$, so $g(x)$ is an antiderivative of f. If F is any other antiderivative of f on $[a, b]$, then F and g differ by a constant: $F(x) = g(x) + C$ for $a < x < b$. Remark that it also holds when $x = a$ and $x = b$.

We put $x = a$ in the formula of $g(x)$ to get

So we have

$$F(b) - F(a) = \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}.$$

□

□ We often use notation $F(x)|_a^b = F(b) - F(a)$.

The Fundamental Theorem of Calculus (page 398)**.** *Suppose f is continuous on $[a, b]$.*

(1) *If $g(x) = \int_a^x f(t)\,\mathrm{d}t$, then $g'(x) = f(x)$.*

(2) $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$, *where F is any antiderivative of f, that is, $F' = f$.*

5.4 Indefinite Integrals and the Net Change Theorem, page 402

Table of Indefinite Integrals (page 403).

$$\int cf(x)\,\mathrm{d}x = c\int f(x)\,\mathrm{d}x \qquad \int (f(x)+g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x$$

$$\int k\,\mathrm{d}x = kx + C$$

$$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$$

$$\int \mathrm{e}^x\,\mathrm{d}x = \mathrm{e}^x + C \qquad \int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,\mathrm{d}x = -\cos x + C \qquad \int \cos x\,\mathrm{d}x = \sin x + C$$

$$\int \sec^2 x\,\mathrm{d}x = \tan x + C \qquad \int \csc^2 x\,\mathrm{d}x = -\cot x + C$$

$$\int \sec x \tan x\,\mathrm{d}x = \sec x + C \qquad \int \csc x \cot x\,\mathrm{d}x = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,\mathrm{d}x = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \sin^{-1} x + C$$

$$\int \sinh x\,\mathrm{d}x = \cosh x + C \qquad \int \cosh x\,\mathrm{d}x = \sinh x + C$$

DtgOpr1BzjE

在微分的階段曾要求各位背熟每個基本函數的導函數，現在要開始熟悉逆操作，必須熟記左方表格中每個函數的反導函數。

因爲不定積分的意義是要找所有的反導函數，而 4.9 的定理 2 告知任兩個反導函數之間只會差一個常數，所以寫不定積分的時候，必須補上 $+C$ 才會代表所有的反導函數。

We adopt the convention that when a formula for a general indefinite integral is given, it is valid only on an interval. For example, the general antiderivative of the function $f(x) = \frac{1}{x}, x \neq 0$ is

$$F(x) = \begin{cases} \ln|x| + C_1 & \text{if } x > 0 \\ \ln|x| + C_2 & \text{if } x < 0. \end{cases}$$

這裡要注意的是 $\int \frac{1}{x}\,\mathrm{d}x$ 的結果，一個是回去看 3.6 例題 2 的介紹；另一個要注意到 $\ln|x|$ 的圖形拆成左右兩半，兩者不相連，所以 $+C$ 實際上在左半與右半可以允許不同的常數。

☐ A definite integral $\int_a^b f(x)\,\mathrm{d}x$ is a number; an indefinite integral $\int f(x)\,\mathrm{d}x$ is a family of functions.

定積分得到的是數字；不定積分指的是一族函數。

Applications

Recall that the Fundamental Theorem of Calculus, part 2:

The Fundamental Theorem of Calculus, Part 2 (page 396). *If f is continuous on $[a, b]$, then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a),$$

where $F(x)$ is any antiderivative of $f(x)$, that is, a function such that $F'(x) = f(x)$.

iEDcEemzZdw

將微積分基本定理第二部份稍做改寫就可得到淨變化定理 (Net Change Theorem)。

We put $f(x) = F'(x)$ into the Theorem and get

Net Change Theorem (page 406)**.** *The integral of a rate of change is the net change:*

$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a).$$

This principle can be applied to all of the rates of change in the natural and social sciences. For example,

積分會有正負相消的效應，所以速度(搭配方向表示正負）的積分是「位移」；若要計算所有路徑的「總距離」，則必須分段調整所有的符號。在計算「面積大小」的時候也同樣要注意這個情況。

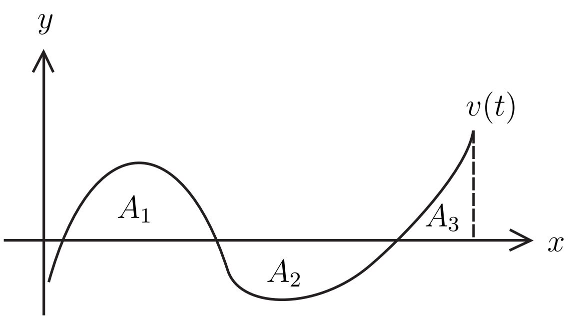
- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

$$\int_{t_1}^{t_2} v(t)\,\mathrm{d}t = s(t_2) - s(t_1)$$

is the net change of position, or *displacement*, of the particle during the time period from t_1 to t_2.

If we want to calculate the distance the object travels during the time interval, we have to consider the intervals when $v(t) \geq 0$ and also the intervals when $v(t) \leq 0$. In both cases the distance is computed by integrating $|v(t)|$, the speed. Therefore,

$$\int_{t_1}^{t_2} |v(t)|\,\mathrm{d}t = \text{total distance traveled.}$$



Displacement =

Distance =

Figure 1: Displacement and distance.

- The acceleration of the object is $a(t) = v'(t)$, so

$$\int_{t_1}^{t_2} a(t)\,\mathrm{d}t = v(t_2) - v(t_1)$$

is the change in velocity from time t_1 to time t_2.

5.5 The Substitution Rule (page 413)

The Substitution Rule (page 413). *If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then*

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(g(x))\,\mathrm{d}g(x) = \int f(u)\,\mathrm{d}u.$$

qaEquBo-8E8

同學不應被這個變數變換定理的型式困惑住，可直接從之後的所有例題與練習，把這個觀念建立即可。

Proof. Suppose F is an antiderivative of f, then we have

$$\frac{\mathrm{d}}{\mathrm{d}x}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x).$$

So $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$. Let $u = g(x)$, then

$$\int f(g(x))g'(x)\,\mathrm{d}x = F(g(x)) + C = F(u) + C = \int F'(u)\,\mathrm{d}u = \int f(u)\,\mathrm{d}u.$$

The middle formula comes from the definition of differential: $\mathrm{d}g(x) = g'(x)\,\mathrm{d}x$. □

The Substitution Rule for Definite Integrals (page 416). *If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then*

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u.$$

Proof. Let F be an antiderivative of f. Then $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$, by Part 2 of the Fundamental Theorem, we have

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = F(g(x))\Big|_a^b = F(g(b)) - F(g(a)).$$

On the other hand, for the right hand side of the equation, we have

$$\int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u = F(u)\Big|_{g(a)}^{g(b)} = F(g(b)) - F((a)).$$

□

Example 1. Compute the integral $\int \frac{x^7}{\sqrt{x^4+1}}\,\mathrm{d}x$.

K6SffXf04IY

根號內的量有點複雜，把它記爲變數 u，再把其它部分全部替換成和 u 有關的量。

Solution.

影片介紹了兩種寫法，個人提倡盡量使用第二種方法，除非函數太複雜，不然在多數情況下是不需要再引進新的符號 u。只要把 $\cos x$ 在心中想成是 u 之後就直接寫答案了，這樣的寫法可以省很多事。

Example 2 (page 410)**.** Calculate $\int \tan x \, \mathrm{d}x$.

Solution.

z7awm81S0g8

善用微分 (differential) 的所有特性，觀察被積分函數，將當中的某一部分轉變到 d 的右邊：$\sec^2\theta \, \mathrm{d}\theta = \mathrm{d}\tan\theta = \mathrm{d}(\tan\theta + 1)$。

Example 3. Find $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sqrt{1+\tan\theta}} \, \mathrm{d}\theta$.

Solution.

善用微分的「線性」，如 $\mathrm{d}x = \frac{1}{a}\mathrm{d}(ax+b)$。

Example 4. Find $\int \cos(ax+b) \, \mathrm{d}x$.

Solution.

$\frac{1}{x}\mathrm{d}x = \mathrm{d}\ln x$，將函數搬到 d 的右邊是進行「積分」。注意到這裡是不需要寫成 $\mathrm{d}\ln|x|$，這是因爲被積分函數有 $(\ln x)^k$，函數有意義的地方只有 $x > 0$ 的部份，所以不需要處理 $x < 0$。

Example 5. Find $\displaystyle\int \frac{(\ln x)^k}{x} \, \mathrm{d}x$.

Solution.

三角函數有很多的恆等式，導致會有些障眼法，若是看得出 $\frac{1}{\sec x}\mathrm{d}x = \cos x \, \mathrm{d}x = \mathrm{d}\sin x$，這個積分問題就變得很容易。

Example 6. Find the integral $\displaystyle\int \frac{\mathrm{e}^{\sin x}}{\sec x} \, \mathrm{d}x$.

Solution.

Example 7. Let $F(x) = \int_0^x (x-t)t\sin(t^2)\,\mathrm{d}t$. Find $F'(x)$.

Solution.

rkN46kWuW6c

這個問題要特別注意變數 x 不僅出現在積分的上限，也出現在被積分的函數中，所以兩個地方都要照顧到。

Example 8. Compute the integral $\displaystyle\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{\sin^{-1}\sqrt{x}}{\sqrt{x(1-x)}}\,\mathrm{d}x$.

Solution.

這個例子就不容易看出變換的關係，就試著逐一替換變數 u，記得積分上、下限也要跟著換。

Integrals of Symmetric Functions (page 417)**.** *Suppose f is continuous on $[-a,a]$.*

(a) *If f is even, then* $\displaystyle\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x$.

(b) *If f is odd, then* $\displaystyle\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$.

eNtFcCxE7xA

當函數 $f(x)$ 是奇函數或偶函數，在 $[-a,a]$ 的定積分會有一些好的性質。這兩個性質從積分的幾何意義來看會覺得非常自然。

而這兩個性質在一些實際的積分問題下很好用，對於偶函數來說，下限是負的數字，帶入反導函數會產生太多負號，很容易被困住; 帶入 $x=0$ 之下，常常會有很多項不見。若能確定函數是奇函數，那就根本不需要去尋找它的反導函數。

Proof. We compute

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{-a}^{0} f(x)\,\mathrm{d}x + \int_0^a f(x)\,\mathrm{d}x = -\int_0^{-a} f(x)\,\mathrm{d}x + \int_0^a f(x)\,\mathrm{d}x,$$

Let $u=-x$, then $\mathrm{d}u = -\mathrm{d}x$ and when $x=-a$, $u=a$. Therefore

$$-\int_0^{-a} f(x)\,\mathrm{d}x = -\int_0^a f(-u)(-\mathrm{d}u) = \int_0^a f(-u)\,\mathrm{d}u.$$

(a) If f is even, then ____________, so we get

$$\int_{-a}^{a} f(x)\,\mathrm{d}x =$$

$$=$$

(b) If f is odd, then ____________, so we get

$$\int_{-a}^{a} f(x)\,\mathrm{d}x =$$

□

Appendix

這個附錄雖然沒有影片的解說，但是這一段話卻道出積分理論的最重要的概念，微積分有沒有把最深刻的精髓學到，就端看各位能不能深刻體會這一段話。

Suppose that $f(x) \in C^1([a,b])$, which implies $|f'(x)| \leq M$. Let $\Delta x = \frac{b-a}{n}$, then

$$\left| \int_a^b f(x)\,\mathrm{d}x - \sum_{i=1}^{n} f(x_i^*)\Delta x \right| \leq \sum_{i=1}^{n} \left| \max_{[x_{i-1},x_i]} f(x) - \min_{[x_{i-1},x_i]} f(x) \right| \Delta x$$

$$\leq \sum_{i=1}^{n} |f'(\xi_i)|(\Delta x)^2 = M \cdot \sum_{i=1}^{n} \frac{(b-a)^2}{n^2} = M \cdot \frac{(b-a)^2}{n} \to 0 \text{ as } n \to \infty.$$

So for integration, before we take summation, the $\frac{1}{n}$ part is the whole material. We can ignore higher order term such as $\frac{1}{n^2}$ because it tends to zero after summation and n tends to infinity.

Therefore, for the Substitution Rule, we only focus on the "differentials" between two variables y and x. That is, $\mathrm{d}y = y'(x)\,\mathrm{d}x$ will catch all information of $\frac{1}{n}$ part between variables y and x.