Chapter 4 Applications of Differentiation

4.1 Maximum and Minimum Values, page 276

oBLCfVT2JF0

Definition 1 (page 276)**.** Let c be a number in the domain D of a function f. Then $f(c)$ is the

(1) *absolute maximum value* (絕對極大值) of f on D if $f(c) \geq f(x)$ for all x in D.

(2) *absolute minimum value* (絕對極小值) of f on D if $f(c) \leq f(x)$ for all x in D.

☐ Absolute maximum (or minimum) 有時候也稱爲 *global maximum* (or *minimum*).

☐ 所有的絕對極大值、絕對極小值統稱爲函數 f 的極值 (*extreme values*).

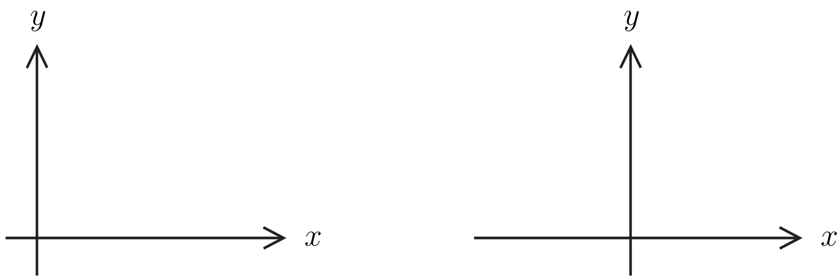

Figure 1: Absolute maximum value and absolute minimum value of f.

☐ 判斷絕對極值時, 必須定義域內「所有」(for all) 的點都要做比較。

Definition 2 (page 276)**.** The number $f(c)$ is a

(1) *local maximum value* (局部極大值) of f on D if $f(c) \geq f(x)$ when x is near c.

(2) *local minimum value* (局部極小值) of f on D if $f(c) \leq f(x)$ when x is near c.

We say that something is true *near* c, we mean that it is true on "some *open* interval containing c."

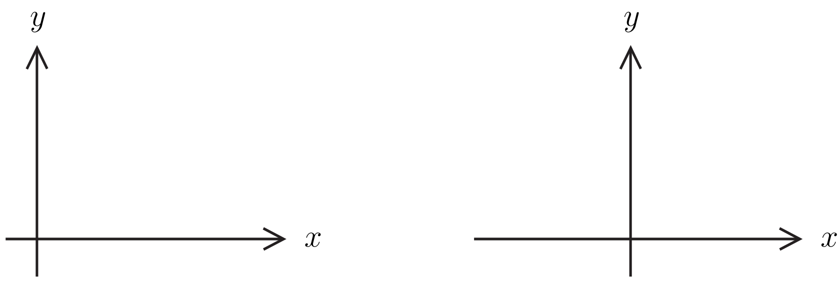

Figure 2: Local maximum value and local minimum value of f.

☐ 局部極值的定義中,「附近」(near) 這個詞很重要。

efbompPs1hs

Example 3. State the absolute (and local) maximum (and minimum) values of the function $y = f(x)$.

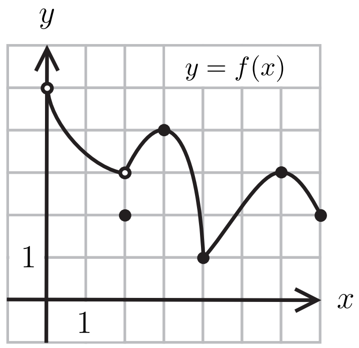

Figure 3: Find absolute (and local) maximum (and minimum) values of the function.

Solution.

(a) Absolute maximum:

(b) Local maximum:

(c) Absolute minimum:

(d) Local minimum:

Theorem 4 (The Extreme Value Theorem, 極值定理, page 278)**.** *If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.*

☐ 極值定理的條件是「閉區間」上的「連續函數」。

☐ 極值定理的結論只告知「存在性」。

Example 5. Give examples that if f is not continuous, or f is continuous on (a, b), the Extreme Value Theorem does not hold. Plot a continuous function that it attains maximum values and minimum values at more than one number.

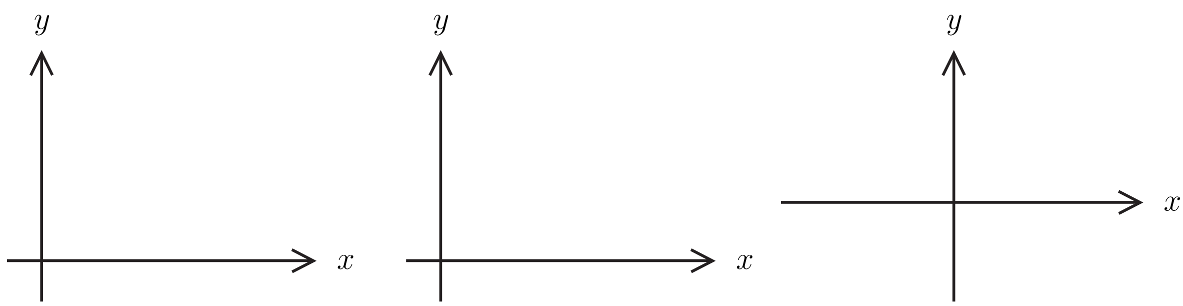

Figure 4: Study the Extreme Value Theorem.

Theorem 6 (Fermat's Theorem 費馬定理, page 279)**.** *If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.*

X3SRnYvweSQ

☐ 條件 "$f'(c)$ exists" 很重要。反例: ______________。

☐ 一般而言, 費馬定理的反敘述不對, 例如: ______________。

Proof. Here we prove the local maximum case. Since $f(c) \geq f(x)$ if x is sufficiently close to c, this implies that if h is sufficiently close to 0, with h being positive or negative, then $f(c) \geq f(c+h)$, or equivalently, $f(c+h) - f(c) \leq 0$.
<u>If $h > 0$</u>, we have $\frac{f(c+h)-f(c)}{h} \leq 0$. Since $f'(c)$ exists, we get

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} =$$

<u>If $h < 0$</u>, we have $\frac{f(c+h)-f(c)}{h} \geq 0$. Since $f'(c)$ exists, we get

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} =$$

Hence $f'(c) = 0$. ☐

Definition 7 (page 280)**.** A *critical number* (臨界點) of a function f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.

TIa3VmnB5ek

☐ 臨界點 (critical numbers) 是在函數 f 的「定義域」內。

Example 8 (page 280)**.** Find the critical numbers of $f(x) = x^{\frac{3}{5}}(4-x) = 4x^{\frac{3}{5}} - x^{\frac{8}{5}}$.

Solution. We compute

$$f'(x) =$$

Therefore the critical numbers are ________.

Theorem 9 (page 280)**.** *If f has a local maximum or minimum at c, then c is a critical number of f.*

☐ 此定理等價於"若函數 f 不存在臨界點, 則 f 沒有局部極大值, 也沒有局部極小值。"

The Closed Interval Method (page 280)**.** *To find the absolute maximum and minimum values of a piecewise continuous function on a closed interval $[a, b]$:*

(1) *Find the values of f at the critical numbers of f in (a, b).*

(2) *Find the values of f at the endpoints of the interval, that is, $f(a)$ and $f(b)$.*

(3) *The largest and smallest of the values from (1) and (2) are absolute maximum value and absolute minimum value, respectively.*

☐ 求絕對極值的方法: 找出所有臨界點與端點; 比較那些點的函數值之大小。

4.2 The Mean Value Theorem, page 287

3R2E7ilyfck

Question 1. A highway from Taipei to Kaohsiung is 330 km and the speed limit is 110 km/h. Man A drove the car on the high way from Taipei at 9 : 00 AM to Kaohsiung at 11 : 59 AM. Did he exceed the speed limit?

Theorem 2 (Rolle's Theorem, page 287)**.** *Let f be a function that satisfies the following three hypotheses:*

(1) *f is continuous on the closed interval $[a, b]$.*

(2) *f is differentiable on the open interval (a, b).*

(3) *$f(a) = f(b)$.*

Then there is a number c in (a, b) such that $f'(c) = 0$.

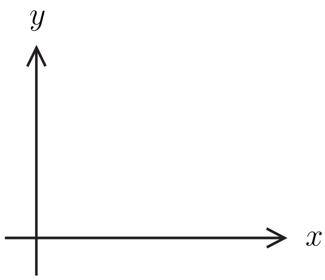

Figure 1: Rolle's Theorem.

Proof. There are three cases.

(I) $f(x) = k$, a constant. We have $f'(x) = 0$, so the number c can be taken to be *any* number in (a, b).

(II) $f(x) > f(a)$ for some x in (a, b). By the ____________________, f has a maximum somewhere in $[a, b]$. Since $f(a) = f(b)$, it must attain this maximum value at a number c in the open interval (a, b). Then f has a ____________ at c, and f is differentiable at c. By _______________, we know $f'(c) = 0$.

(III) $f(x) < f(a)$ for some x in (a, b). By the ____________________, f has a minimum value in $[a, b]$, and since $f(a) = f(b)$, it attains this local minimum value at a number $c \in (a, b)$. By _______________, $f'(c) = 0$.

□

☐ 定理條件, 函數 $f(x)$ 必須在「閉區間連續」。

☐ 定理條件, 函數 $f(x)$ 必須在開區間「可微分」(每一個點)。

☐ 定理結論只告知「存在性」。

Example 3. Give examples that each condition in Rolle's Theorem is required.

x61RgE7n8o4

Solution.

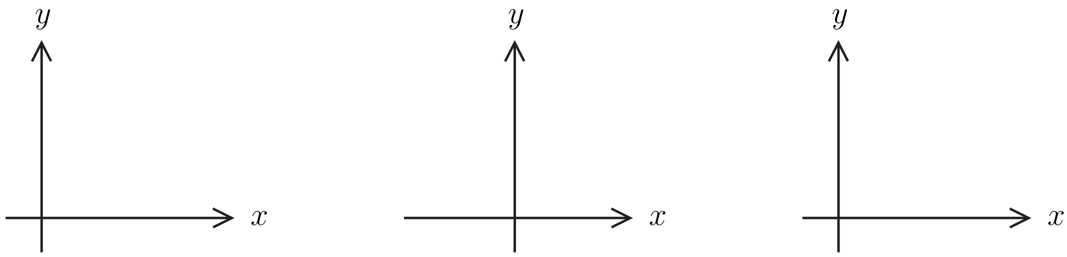

Figure 2: Study Rolle's Theorem.

Example 4 (page 287)**.** Prove that $x^3 + x - 1 = 0$ has exactly one real root.

Solution.

Theorem 5 (The Mean Value Theorem, 平均值定理, page 288)**.** *Let f be a function that satisfies the following hypotheses:*

ZtR_UIq3vxQ

(1) *f is continuous on the closed interval $[a, b]$.*

(2) *f is differentiable on the open interval (a, b).*

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \textit{or equivalently,} \quad f(b) - f(a) = f'(c)(b - a).$$

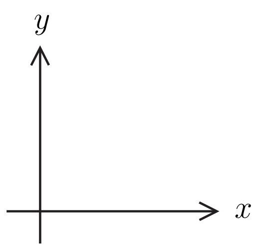

Figure 3: The Mean Value Theorem.

Proof of Mean Value Theorem. Define a new function

$$h(x) =$$

We will verify that $h(x)$ satisfies the three hypotheses of Rolle's Theorem.

(1) The function h is continuous on $[a, b]$: It is the sum of f and a first-degree polynomial, both of which are continuous.

(2) The function h is differentiable on (a, b): Both f and the first-degree polynomial are differentiable. In fact, we have

$$h'(x) =$$

(3) $h(a) = h(b) = 0$:

$$h(a) =$$

$$h(b) =$$

By ______________, there is a number $c \in (a, b)$ such that $h'(c) = 0$. Therefore,

□

□ 均值定理也是要注意「連續」、「可微」、「存在」。

□ 爲什麼這個定理要叫做「平均值定理」?

58cSDtk3T9M

Theorem 6. *If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).*

Proof. Let x_1 and x_2 be any two numbers in (a, b) with $x_1 < x_2$. Since f is differentiable on (a, b), it must be differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$. By applying the __________________ to f on the interval $[x_1, x_2]$, we get a number c such that $x_1 < c < x_2$ and

Therefore f has the same value at *any* two numbers x_1 and x_2 in (a, b). So $f(x)$ is constant on (a, b). □

□ **Theorem 6** 提供一個刻劃常數函數的方法。

Corollary 7. *If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.*

Proof. Let $F(x)$

$\square$

$\square$ 注意 $f(x) = \frac{x}{|x|}$ 與 $g(x) = 1$, 它在 $x \in (-1, 1)$ 當中並不滿足推論當中的條件。

Example 8. Show that $\left|\tan\frac{x}{2} - \tan\frac{y}{2}\right| \geq \frac{|x-y|}{2}$ for any $x, y \in (-\pi, \pi)$.

WjWwBQAaW3Q

Solution. If $x = y$, the inequality holds. If $x \neq y$, without loss of generality, we assume $-\pi < x < y < \pi$. Consider the function $f(t) = \tan\frac{t}{2}$, then

- $f(t)$ is ____________________.
- $f(t)$ is ______________________.

By the __________________, there is a number $c \in (x, y)$ such that $f(x) - f(y) = f'(c)(x - y)$, which implies $|f(x) - f(y)| = |f'(c)||x - y|$. Since $f'(t) =$ ______, we have $|f'(c)| =$ __________. So $|f(x) - f(y)| \geq \frac{1}{2}|x - y|$, which means

$$\left|\tan\frac{x}{2} - \tan\frac{y}{2}\right| \geq \frac{|x-y|}{2}.$$

Theorem 9 (Cauchy's Mean Value Theorem, (柯西均值定理) Appendix F, A45)**.** *Suppose that the functions f and g are continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ for all x in (a, b). Then there is a number $c \in (a, b)$ such that*

biygd5TQf38

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. The key point is to find a new function $F(x)$ and apply the Mean Value Theorem.

$$F(x) =$$

$\square$

4.4 Indeterminate Forms and l'Hospital's Rule, page 304

PwmK2Ha2SAA

In this section, we want to introduce a new method to deal with the limit such as

$$\lim_{x\to\infty}\frac{\ln x}{x-1}, \quad \text{or} \quad \lim_{x\to\infty}\frac{x^2}{e^x}.$$

Definition 1 (page 304–305)**.**

(1) If we have a limit of the form $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$, where both $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$, it is called an *indeterminate form of type* $\frac{0}{0}$ (零分之零的不定型).

(2) If we have a limit of the form $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$, where both $f(x)\to\infty$ (or $-\infty$) and $g(x)\to\infty$ (or $-\infty$) as $x\to a$, it is called an *indeterminate form of type* $\frac{\infty}{\infty}$ (無限大分之無限大的不定型).

L' Hospital's Rule (page 305)**.** *Suppose f and g are differentiable and $g'(x)\neq 0$ on an open interval I that contains a (except possibly at a). Suppose that a limit has an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

- ☐ 務必檢查定理的條件: (1) 是否爲不定型; (2) 檢查 $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ 是否存在。
- ☐ 定理也適用於單邊極限。
- ☐ 可以串聯至有限次可微分函數。$\lim\limits_{x\to a}\frac{f(x)}{g(x)}\overset{L}{=}\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}\overset{L}{=}\cdots\overset{L}{=}\lim\limits_{x\to a}\frac{f^{(k)}(x)}{g^{(k)}(x)}=M.$

aBKa8s0dzCM

Example 2.

(a) Find $\lim\limits_{t\to 0^+}\frac{t-\ln(1+t)}{t^2}$.

(b) Use (a) to find $\lim\limits_{t\to 0^+}\frac{\sqrt{t-\ln(1+t)}}{t}$.

Solution.

Indeterminate Products, page 308

Definition 3 (page 305)**.** If we have a limit of the form $\lim_{x\to a} f(x)g(x)$, where $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$) as $x \to a$, it is called an *indeterminate form of type* $0 \cdot \infty$. (零乘以無限大的不定型)

VdDFNTkTC1k

We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f},$$

and this converts the given limit into an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 4 (page 308)**.** Evaluate $\lim_{x\to 0^+} x \ln x$.

Solution.

☐ 如何把函數分配至分子或分母是一門學問。

Indeterminate Differences, page 309

Definition 5 (page 305)**.** If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

wbQEYnzfK1Q

$$\lim_{x\to a}(f(x) - g(x))$$

is called an *indeterminate form of type* $\infty - \infty$ (無限大減無限大的不定型).

We can try to convert the difference into a quotient (for instance, by using a common denominator (通分), or rationalization (有理化), or factoring out a common factor (提公因式)) so that we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 6. Find the limit $\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$.

Solution.

☐ 隨時觀察並恰當整理函數, 不要盲目使用 l'Hospital Rule。

Indeterminate Powers, page 310

h2ifbDam0Xw

Definition 7 (page 310)**.** Several indeterminate forms arise from the limit

$$\lim_{x\to a}(f(x))^{g(x)}$$

(1) $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$: type 0^0.

(2) $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$: type ∞^0.

(3) $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$: type 1^∞.

Each of these three cases can be treated either by taking the natural logarithm: let $y = (f(x))^{g(x)}$, then $\ln y = g(x)\ln f(x)$ or by writing the function as an exponential: $(f(x))^{g(x)} = \mathrm{e}^{g(x)\ln f(x)}$. In either method we are led to the indeterminate product $g(x)\ln f(x)$, which is of type $0\cdot\infty$.

Example 8 (page 310)**.** Find $\lim_{x\to 0^+} x^x$.

Solution.

☐ 取對數算出極限值後, 記得還原。

幾個使用 l'Hospital Rule 的經驗

o_OcpMZYcps

- 使用前務必檢查定理的條件: (1) 是否爲不定型; (2) 檢查 $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ 是否存在。

$$\lim_{x\to 0}\frac{x}{1+\sin x}$$

$$\lim_{x\to \infty}\frac{x-\sin x}{x+\sin x}$$

- 避免鬼打牆; 像是 $\sin x, \cos x$ (as $x\to\infty$) 或是 $\sin\frac{1}{x}, \cos\frac{1}{x}, \frac{1}{x}, \frac{1}{\ln x}$ (as $x\to 0$)。

$$\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x} \overset{(\frac{0}{0}),L}{=} \lim_{x\to 0}\frac{2x\sin\frac{1}{x}+x^2\cos\frac{1}{x}(-\frac{1}{x^2})}{\cos x} = \lim_{x\to 0}\frac{2x\sin\frac{1}{x}-\cos\frac{1}{x}}{\cos x}\ldots?$$

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{x}{\frac{1}{\ln x}} \overset{(\frac{0}{0}),L'}{=} \lim_{x\to 0^+}\frac{1}{-\frac{1}{(\ln x)^2}\cdot\frac{1}{x}} = \lim_{x\to 0^+} -x(\ln x)^2\ldots \text{ 鬼打牆}$$

- 分子分母適時地整理、重新分配, 或是變數變換, 有助於計算。

$$\lim_{x\to 0^+}\frac{1}{x^2 e^{\frac{1}{x}}} = \lim_{x\to 0^+}\frac{e^{-\frac{1}{x}}}{x^2} \overset{(\frac{0}{0}),L'}{=} \lim_{x\to 0^+}\frac{e^{-\frac{1}{x}}\cdot\frac{1}{x^2}}{2x} = \lim_{x\to 0^+}\frac{e^{-\frac{1}{x}}}{2x^3}\ldots \text{ 鬼打牆}$$

$$\lim_{x\to 0^+}\frac{1}{x^2 e^{\frac{1}{x}}} = \lim_{x\to 0^+}\frac{\frac{1}{x^2}}{e^{\frac{1}{x}}} \overset{(\frac{\infty}{\infty}),L}{=} \lim_{x\to 0^+}\frac{-2\frac{1}{x^3}}{e^{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0^+}\frac{2}{x e^{\frac{1}{x}}}\ldots \text{ 情況變好, 再用一次}$$

$$\lim_{x\to 0^+}\frac{1}{x^2 e^{\frac{1}{x}}}$$

$$\lim_{x\to 0}\frac{\tan x - x}{x-\sin x}$$

$$\lim_{x\to 0^+}\frac{\tan 3x}{\sqrt{1-\cos 2x}}$$

- l'Hospital Rule 某種程度是"大絕", 但並非萬能。

$$\lim_{x\to \infty}\frac{(\sin x)e^x}{(x+\sin x)e^{2x}}$$

- 記得其他求極限的方法, 像是 Squeeze Theorem, definition of derivative, 也很好用。(之後還會介紹用積分方法求極限, 下學期會介紹使用泰勒展式法求極限。)

$$\lim_{x\to 0}\frac{\sin x}{x} = \_\_\_ \qquad \lim_{x\to\infty} x\sin\frac{1}{x} = \_\_\_ \qquad \lim_{x\to\infty}\frac{\sin x}{x} = \_\_\_ \qquad \lim_{x\to 0} x\sin\frac{1}{x} = \_\_\_$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \_\_\_ \qquad \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \_\_\_ \qquad \lim_{x\to 0}\frac{\ln(1+x)}{x} = \_\_\_$$

$$\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x = \_\_\_ \qquad \lim_{x\to\infty}(1+x)^{\frac{1}{x}} = \_\_\_ \qquad \lim_{x\to\infty}\frac{\ln(1+x)}{x} = \_\_\_$$

Appendix

Proof of l'Hospital's Rule (Appendix A46)

EH5tz1psN8c

We are assuming that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. Let $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$. Define

$$F(x) = \begin{cases} f(x) & \text{if } x \neq a \\ 0 & \text{if } x = 1 \end{cases}, \qquad G(x) = \begin{cases} g(x) & \text{if } x \neq a \\ 0 & \text{if } x = 1 \end{cases}.$$

Then both F and G are continuous on I since $f$and g are continuous on $\{x \in I | x \neq a\}$ and

$$\lim_{x\to a} F(x) = \lim_{x\to a} f(x) = 0 = F(a), \qquad \lim_{x\to a} G(x) = \lim_{x\to a} g(x) = 0 = G(a).$$

Furthermore, F and G are differentiable on (a, x) (or (x, a)) since $F' = f'$ and $G' = g'$. Since $G' \neq 0$, by the Cauchy's Mean Value Theorem, there is a number y such that $a < y < x$ (or $x < y < a$) and

$$\frac{F'(y)}{G'(y)} = \frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F(x)}{G(x)}.$$

Hence

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{F(x)}{G(x)} = \lim_{y\to a^+} \frac{F'(y)}{G'(y)} = \lim_{y\to a^+} \frac{f'(y)}{g'(y)} = L,$$
$$\left(\text{and} \quad \lim_{x\to a^-} \frac{f(x)}{g(x)} = \lim_{x\to a^-} \frac{F(x)}{G(x)} = \lim_{y\to a^-} \frac{F'(y)}{G'(y)} = \lim_{y\to a^-} \frac{f'(y)}{g'(y)} = L.\right)$$

Therefore,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

This proves l'Hospital's Rule for the case where a is finite.

If a is infinite, we let $t = \frac{1}{x}$. Then $t \to 0^+$ as $x \to \infty$, so we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0^+} \frac{f(\frac{1}{t})}{g(\frac{1}{t})} \overset{L}{=} \lim_{t\to 0^+} \frac{f'(\frac{1}{t}) \cdot \left(-\frac{1}{t^2}\right)}{g'(\frac{1}{t}) \cdot \left(-\frac{1}{t^2}\right)} = \lim_{t\to 0^+} \frac{f'(\frac{1}{t})}{g'(\frac{1}{t})} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

L'Hospital Rule 與極限法則的合併使用

回想極限的四則運算法則與羅必達法則:

Limit Laws (page 99)**.** *Suppose that* $\lim\limits_{x\to a} p(x)$ *and* $\lim\limits_{x\to a} q(x)$ *exist. Then*

$$\lim_{x\to a}\frac{p(x)}{q(x)} = \frac{\lim\limits_{x\to a} p(x)}{\lim\limits_{x\to a} q(x)} \quad \text{if} \quad \lim_{x\to a} q(x) \neq 0.$$

L' Hospital's Rule (page 302)**.** *Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that*

$$\lim_{x\to a} f(x) = 0 \quad \textit{and} \quad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \quad \textit{and} \quad \lim_{x\to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

這兩個定理可以搭配起來靈活運用, 比方說:

Theorem 9. *Suppose that $p(x), q(x)$ satisfy the assumptions of Limit Laws with* $\underline{\lim\limits_{x\to a} p(x) \neq 0}$*, and suppose $f(x), g(x)$ satisfy the assumptions of L'Hospital Rule. Then*

$$\lim_{x\to a}\frac{p(x)f(x)}{q(x)g(x)} = \lim_{x\to a}\frac{p(x)}{q(x)} \cdot \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

上述定理要強調的是說: 雖然分子 $p(x)f(x)$ 整體看取極限是 0, 分母 $q(x)g(x)$ 整體看取極限是 0, 於是 $\frac{p(x)f(x)}{q(x)g(x)}$ 是 $\frac{0}{0}$ 的不定型, 但是當分子與分母各自只有一部分是不定型 (只有 $\frac{f}{g}$ 是 $\frac{0}{0}$), 而 <u>p, q 具有非零的極限</u>, 那麼就可以把 p, q 的極限抽出來, 考慮剩下的不定型之極限, 再相乘。

理由很簡單, 因爲 $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)} = L$, 所以下式的第一個等式合法 (乘法法則)

$$\lim_{x\to a}\frac{p(x)f(x)}{q(x)g(x)} = \lim_{x\to a}\frac{p(x)}{q(x)} \cdot \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{p(x)}{q(x)} \cdot \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

定理一就是大大化簡計算的一個方法, 因爲有很多人一看到 $\frac{0}{0}$ 就不加思索的地上下微分(典型的"看見黑影就開槍"), 雖然還是可以算出答案, 但是過程中可能會添上很多計算上的麻煩。

4.3 How Derivatives Affect the Shape of a Graph, page 293

8H-9SUDxCqo

Increasing/Decreasing Test (page 293)**.**

(a) *If $f'(x) > 0$ on an interval, then f is increasing on that interval.*

(b) *If $f'(x) < 0$ on an interval, then f is decreasing on that interval.*

Proof.

(a) Let $x_1 < x_2$. By the ____________________, there is $c \in (x_1, x_2)$ such that

.

(b) Let $x_1 < x_2$. By the ____________________, there is $c \in (x_1, x_2)$ such that

.

□

Example 1. Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Solution. We compute $f'(x) =$

Solutions of $f'(x) = 0$ are ____________. Hence

$f(x)$ is increasing on ________________; $f(x)$ is decreasing on ________________.

ehahVquKqMk

The First Derivative Test (page 294)**.** *Suppose that c is a critical number of a continuous function f.*

(a) *If f' changes from positive to negative at c, then f has a local maximum at c.*

(b) *If f' changes from negative to positive at c, then f has a local minimum at c.*

(c) *If f' does not change sign at c (for example, if f' is positive on both side of c or negative on both sides), then f has no local maximum or minimum at c.*

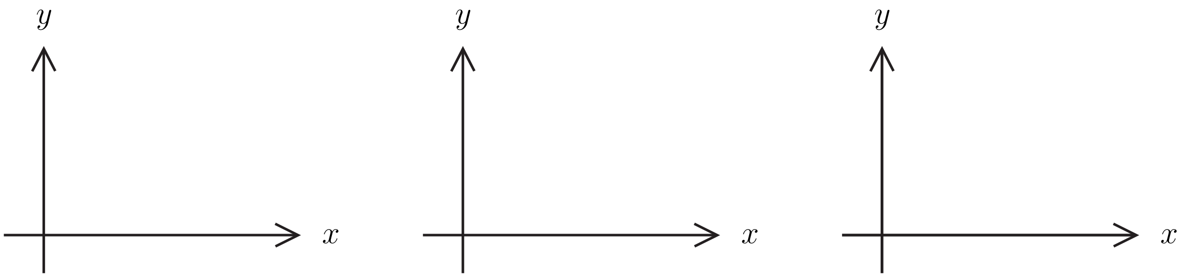

Figure 1: The First Derivative Test.

Example 2. Find the local minimum and maximum values of the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ in **Example 1**.

Solution.

x	-1	0	2
f	0	5	-27
f'			

Hence f has local maximum ________; f has local minimum ____________________.

Definition 3 (page 296)**.** If the graph f lies above all of it tangents on an interval I, then it is called *concave upward* (凹口朝上) on I. If the graph f lies below all of it tangents on an interval I, then it is called *concave downward* (凹口朝下) on I.

9-0b-5pEFvM

☐ 有些教科書或文獻使用凸函數 (convex) 取代凹口向上 (concave upward)。

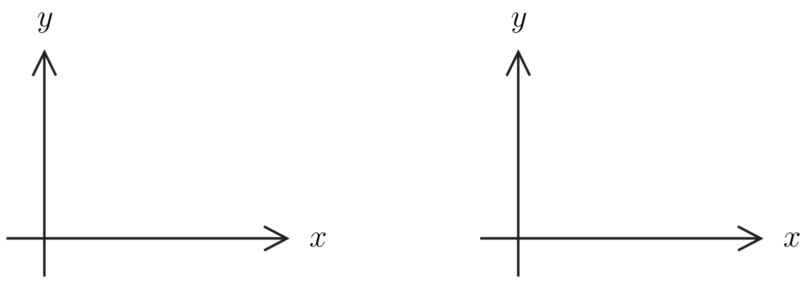

Figure 2: Concave upward and concave downward.

Concavity Test (page 296)**.**

(a) *If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.*

(b) *If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.*

Proof of (a). Since $f''(x) > 0$ in I, we know that $f'(x)$ is increasing in I. Given $x_0 \in I$, the tangent line equation to the graph of $f(x)$ at $(x_0, f(x_0))$ is

$$y - f(x_0) = f'(x_0)(x - x_0) \Rightarrow y = f'(x_0)(x - x_0) + f(x_0).$$

We will show that $f(x) \geq f'(x_0)(x - x_0) + f(x_0)$ for all $x \in I$.

Consider the function

$$F(x) = f(x) - f'(x_0)(x - x_0) - f(x_0) \quad \text{for} \quad x \in I.$$

First, we know that $F(x_0) = 0$. Next, we compute $F'(x) = f'(x) - f'(x_0)$, which implies $F'(x_0) = f'(x_0) - f'(x_0) = 0$. Since $F'(x) < 0$ for $x < x_0$ and $F'(x) > 0$ for $x > x_0$, we know that $F(x_0)$ is a local (and hence absolute) minimum at $x = x_0$ in I. That means $F(x) \geq 0$ for all $x \in I$, thus $f(x) \geq f'(x_0)(x - x_0) + f(x_0)$ for all $x \in I$. ☐

505tEz70tPc

Definition 4 (page 297)**.** A point P on a curve $y = f(x)$ is called an *inflection point* (反曲點) if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

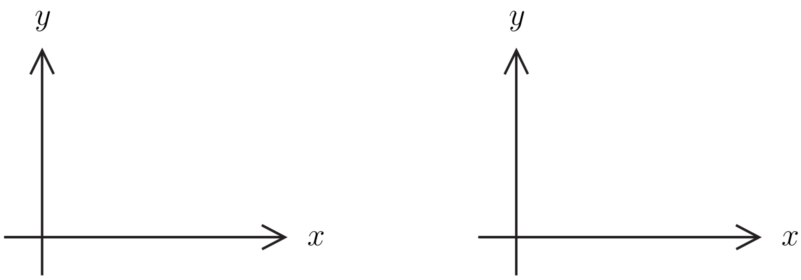

Figure 3: Inflection points.

Example 5. Find the concave upward and downward intervals, and inflection points of the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ in **Example 1**. Sketch the graph of f.

Solution. We compute

$$f''(x) = \qquad\qquad\qquad\qquad\qquad\qquad .$$

So

x		-1	x_1	0	x_2	2	
f		0	$f(x_1)$	5	$f(x_2)$	-27	
f'	$-$	0	$+$	0	$-$	0	$+$
f''							

The points of inflections are ____________________.

f is concave upward on ____________________.

f is concave downward on __________.

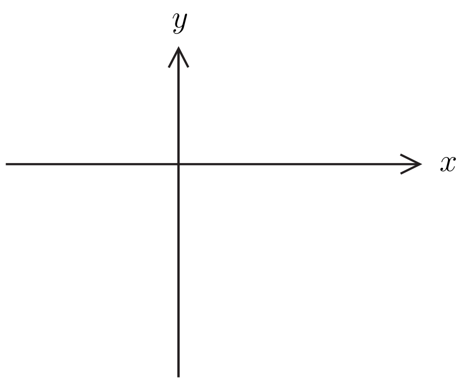

Figure 4: The graph of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

The Second Derivative Test (page 297)**.** *Suppose f'' is continuous near c.*

EU_LBaZWe4k

(a) *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.*

(b) *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.*

Example 6. Show that $f(x) = \frac{\sin x}{x}$ is decreasing on $(0, \frac{\pi}{2})$.

Solution.

☐ 比較 Section 2.3, 那時候證明了 $|\sin x| \leq |x|$。

Example 7. Classify all cubic functions $f(x) = ax^3 + bx^2 + cx + d$.

jFFuiBKJP4M

Solution.

4.5 Summary of Curve Sketching, page 315

bA12vd64GN8

Definition 1 (page 320)**.** If

$$\lim_{x\to\infty}(f(x)-(mx+b))=0,$$

where $m\neq 0$, then the line $y=mx+b$ is called a *slant asymptote* (斜漸近線).

Proposition 2. *The graph of $f(x)$ has a slant asymptote if and only if*

$$\lim_{x\to\infty}\frac{f(x)}{x}=m\neq 0 \quad \textit{and} \quad \lim_{x\to\infty}(f(x)-mx)=b.$$

Proof. When $x>0$,

$$\frac{f(x)}{x}=\frac{f(x)-(mx+b)}{x}+m+\frac{b}{x}\Rightarrow \lim_{x\to\infty}\frac{f(x)}{x}=0+m+0=m.$$

$$\begin{aligned}\lim_{x\to\infty}(f(x)-mx)&=\lim_{x\to\infty}(f(x)-(mx+b)+b)\\&=\lim_{x\to\infty}(f(x)-(mx+b))+\lim_{x\to\infty}b=0+b=b.\end{aligned}$$

Conversely, we have

$$\begin{aligned}\lim_{x\to\infty}(f(x)-(mx+b))&=\lim_{x\to\infty}((f(x)-mx)-b)\\&=\lim_{x\to\infty}(f(x)-mx)-\lim_{x\to\infty}b=b-b=0.\end{aligned}$$

□

- □ 函數圖形當 $x\to-\infty$ 也可能存在斜漸近線, 定義與其等價條件都改成 $\lim\limits_{x\to-\infty}$。
- □ 兩個極限都要存在才能稱函數有斜漸近線。例如 $f(x)=\ln x$ 沒有斜漸近線。

Guidelines for sketching a curve

FBWCqsr2iyk

定 **Domain**: the set of x for which $f(x)$ is defined.

交 **Intercepts**: y-intercept $f(0)$, x-intercepts: let $y=0$ and solve for x.

對 **Symmetry**: even function, odd function, periodic function.

漸 **Asymptotes**: horizontal asymptotes, vertical asymptotes, slant asymptotes.

一 **Intervals of increase or decrease**: use the Increasing/Decreasing test.

極 **Local maximum and minimum values**: find the critical numbers of f ($f'(c)=0$ or $f'(c)$ does not exist.)

二 **Concavity and points of inflection**: compute $f''(x)$ and use the Concavity Test.

圖 **Sketch the Curve**: use the information in items 1–7, draw the graph.

Example 1 (page 317)**.** Sketch the curve $y = f(x) = \frac{2x^2}{x^2-1}$.

eI0fbbIgRy0

Solution.

A. The domain is ______________________.

B. The x- and y-intercept are both _.

C. Since ___________, the function f is ___.

D. Since

$$\lim_{x\to\pm\infty} \frac{2x^2}{x^2-1} = \underline{\qquad\qquad\qquad},$$

the line _____ is a _________________. The denominator is 0 when ______. we compute the following limits:

$$\lim_{x\to 1^+} \frac{2x^2}{x^2-1} = \qquad\qquad \lim_{x\to 1^-} \frac{2x^2}{x^2-1} =$$

$$\lim_{x\to -1^+} \frac{2x^2}{x^2-1} = \qquad\qquad \lim_{x\to -1^-} \frac{2x^2}{x^2-1} =$$

Therefore the lines _____ and ______ are vertical asymptotes.

E. Direct computation gives

$$y' = \underline{\qquad\qquad\qquad\qquad\qquad\qquad}.$$

Since $f'(x) > 0$ when _______________ and $f'(x) < 0$ when ______________, f is increasing on ________________ and decreasing on ______________.

F. The only critical number is _____. Since f' changes from positive to negative at 0, $f(0) = 0$ is a _____________ by the First Derivative Test.

G. Direct computation gives

$$f''(x) = \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}.$$

We know $f''(x) > 0$ on ______ and $f''(x) < 0$ on ______. Thus the curve is concave upward on the interval _________________ and concave downward on ______. It has no point of inflection since ______________________________.

H. Using this information to sketch the curve. (畫在右上角)

3W1Yt_jtLQg

Example 2 (page 317)**.** Sketch the curve $y = f(x) = \frac{x^2}{\sqrt{x+1}}$.

Solution.

A. The domain is _______.

B. The x- and y-intercept are both ___.

C. Symmetry: None.

D. Since

$$\lim_{x\to\infty} \frac{x^2}{\sqrt{x+1}} = \underline{\qquad},$$

there is no horizontal asymptote. Since

$$\lim_{x\to -1^+} \frac{x^2}{\sqrt{x+1}} = \underline{\qquad},$$

the line ______ is a vertical asymptotes.

E. Direct computation gives

$$y' = \underline{\qquad\qquad\qquad\qquad\qquad\qquad}.$$

We see that $f'(x) = 0$ when _____, so the only critical number is _. Since $f'(x) > 0$ when _____ and $f'(x) < 0$ when __________, f is increasing on _____ and decreasing on ______.

F. Since $f'(0) = 0$ and f' changes from negative to positive at 0, $f(0) = 0$ is a __________________________ by the First Derivative Test.

G. Direct computation gives

$$f''(x) = \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}.$$

Since the numerator is always _______, we know $f''(x) > 0$ for all x in the domain of f, which means f is concave upward on _______ and there is no point of inflection.

H. Using this information to sketch the curve. (畫在右上角)

Example 3 (page 318)**.** Sketch the curve $y = f(x) = xe^x$.

1Qp4US5GRcA

Solution.

A. The domain is ___.

B. The x- and y-intercept are both ___.

C. Symmetry: None.

D. Since

$$\lim_{x\to\infty} xe^x = \underline{\qquad},$$

there is no horizontal asymptote. By the l'Hospital Rule, we have

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \underline{\qquad\qquad\qquad\qquad},$$

so the _____ is a horizontal asymptote.

E. Direct computation gives

$$y' = \underline{\qquad\qquad\qquad\qquad\qquad}.$$

Since $f'(x) > 0$ when ______ and $f'(x) < 0$ when ______, f is increasing on ______ and decreasing on ________.

F. Since $f'(-1) = 0$ and f' changes from negative to positive at $x = -1$, $f(-1) = -e^{-1}$ is a __________________________ by the First Derivative Test.

G. Direct computation gives

$$f''(x) = \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}.$$

Since $f''(x) > 0$ if ______ and $f''(x) < 0$ if ______, f is concave upward on ______ and concave downward on ________. The inflection point is __________.

H. Using this information to sketch the curve.

wbkisiSCwIU

Example 4 (page 319)**.** Sketch the curve $y = f(x) = \frac{\cos x}{2+\sin x}$.

Solution.

A. The domain is ____.

B. The x-intercepts are ________________ and y-intercept is ________.

C. Symmetry: f is neither even nor odd. Since $f(x+2\pi) = f(x)$ for all x, f is _______ and has period __. Thus, the following steps we only consider $0 \le x \le 2\pi$ and then extend the curve by translation.

D. Asymptotes: None.

E. Direct computation gives

$$y' = \underline{\hspace{20em}}.$$

Thus $f'(x) > 0$ when ____. So f is increasing on ________ and decreasing on __________________.

F. From part **E** and First Derivative Test, the local minimum value is ____________ and local maximum value is ___________.

G. Direct computation gives

$$f''(x) =$$

Since $f''(x) > 0$ if __________, f is concave upward on ______ and concave downward on ________________. The inflection point is _______________.

H. Using this information to sketch the curve.

Example 5 (page 319)**.** Sketch the curve $y = f(x) = \ln(4 - x^2)$.

NwfDfOYvDcc

Solution.

A. The domain is ______.

B. The y-intercept is $f(0) = \ln 4$. To find the x-intercept, we set $\ln(4 - x^2) = 0$, so we have ____________________. Therefore the x-intercepts are ____.

C. Since $f(-x) = f(x)$, f is ____ and the curve is symmetric about the _____.

D. Since

$$\lim_{x \to -2^+} \ln(4 - x^2) = \underline{\qquad}, \qquad \lim_{x \to 2^-} \ln(4 - x^2) = \underline{\qquad},$$

the lines ________________ are vertical asymptotes.

E. Direct computation gives

$$y' = \frac{\qquad}{\underline{\qquad}}.$$

Since $f'(x) > 0$ when __________ and $f'(x) < 0$ when ________, f is increasing on ______ and decreasing on ____.

F. The only critical number is _____. Since f' changes from positive to negative at 0, $f(0) = \ln 4$ is a ___________________________ by the First Derivative Test.

G. Direct computation gives

$$f''(x) = \frac{\qquad}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}}.$$

Since $f''(x) < 0$ for all x, the curve is ________________ on ______ and has no inflection point.

H. Using this information to sketch the curve.

aoV6RJZFQ2Y

Example 6 (page 320)**.** Sketch the curve $y = f(x) = \frac{x^3}{x^2+1}$.

Solution.

A. The domain is ____.

B. The x- and y-intercept are both ___.

C. Since ______________, the function f is _____.

D. Since $x^2 + 1$ is never 0, there is no vertical asymptote. Since $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, there is no horizontal asymptote. Long division gives

$$f(x) = \frac{x^3}{x^2+1} = \underline{\qquad\qquad\qquad},$$

$$f(x) - x = -\frac{x}{x^2+1} = \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}.$$

So the line _____ is a ______________.

E. Direct computation gives

$$y' = \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}.$$

Since $f'(x) > 0$ when ________________, f is increasing on ________.

F. Although $f'(0) = 0$, f' does not change sign at 0, so there is ________________ or ________.

G. Direct computation gives

$$f''(x) = \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}.$$

Since $f''(x) = 0$ when ______________, we set up the following chart.

The points of inflection are ___________________________.

H. Using this information to sketch the curve.

4.7 Optimization Problems, page 330

Steps in solving optimization problems

1. **Understand the problem**: What is the unknown? What are the given quantities? What are the given conditions?

2. **Draw a diagram**: In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.

3. **Introduce notation**: Assign a symbol to the quantity that is to be maximized or minimized (call it Q for now). Also select symbols $a, b, c, \ldots, x, y$ for other known quantities and label the diagram with these symbols.

4. Express Q in terms of some of the other symbols.

5. If Q has been expressed as a function of more than one variable, use the given information to find relationships among these variables. Then use these equations to eliminate all but one of the variables. Thus we get $Q = f(x)$.

6. Use the methods of Section 4.1 and 4.3 to find the absolute maximum or minimum value of f.

Example 1 (Snell's Law, 斯乃爾定律, page 268)**.** Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to Fermat's Principle, a ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimizes the time taken. Show that kZ6gWTJvVNY

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2},$$

where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are known.

Solution.

eQpoM26CaM4

Example 2 (搬家俱, page 268)**.** A steel pipe is being carried down a hallway a m wide. At the end of the hall there is a right-angled turn into a narrower hallway b m wide. What is the length of the longest pipe that can be carried horizontally around the corner?

Solution.

t2MdXHcMYZI

Example 3 (看畫、教室的風水, page 269)**.** A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer. How far from the wall should the observer stand to get the best view? (In other words, where should the observer stand so as to maximize the angle θ subtended at his eye by the painting?)

Solution.

Example 4. A right circular cone is inscribed in a sphere of radius r. Find the largest possible volume of such a cone. In this case, what is the height and radius of the cone?

hbKrEiTBiqA

Solution.

Example 5 (折紙問題, page 269)**.** The upper right-hand corner of a piece of paper, 30 cm by 20 cm, is folded over to the bottom edge. How would you fold it so as to minimize the length of the fold? In other words, how would you choose x to minimize y?

FzTOC_OCRCO

Solution.

N-KwPosBmfE

Example 6.

(a) Find the point (denote P) on the line $y = x^2$ that is closest to the point $Q(3, 0)$.

(b) Show that the line PQ is orthogonal to the tangent line of $y = x^2$ at P.

Solution.

CzCCC_7mTJg

Example 7 (電影的極佳位置)**.**

4.9 Antiderivative, page 350

Definition 1 (page 350)**.** A function F is called an *antiderivative* (反導函數) of f on an interval I if $F'(x) = f(x)$ for all x in I.

3ES4KjEu5gU

Theorem 2 (page 351)**.** *If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is*

$$F(x) + C,$$

where C is an arbitrary constant.

Proof. If F and G are any two antiderivative of f, then $F'(x) = f(x) = G'(x)$. Form the corollary of the Mean Value Theorem (Section 4.2 Corollary 8), we know $G(x) - F(x) = C$, where C is a constant. So $G(x) = F(x) + C$. □

This is a table of antidifferentiation formulas. We use the notation $F'(x) = f(x)$ and $G'(x) = g(x)$.

gwp9cLFo8Ac

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sec^2 x$	$\tan x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec x \tan x$	$\sec x$
$x^n (n \neq -1)$	$\frac{x^{n+1}}{n+1}$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{1}{x}$	$\ln \lvert x \rvert$	$\frac{1}{1+x^2}$	$\tan^{-1} x$
e^x	e^x	$\cosh x$	$\sinh x$
$\cos x$	$\sin x$	$\sinh x$	$\cosh x$
$\sin x$	$-\cos x$		

Example 3. Find the most general antiderivative of the function. (Let $F(x)$ is the antiderivative of the function $f(x)$.)

(1) $f(x) = \mathrm{e}^2$ $F(x) =$

(2) $f(x) = x(2-x)^2$

$F(x) =$

(3) $f(x) = x^\pi - x^{3.14}$ $F(x) =$

(4) $f(x) = \frac{2+x^2}{1+x^2}$

$F(x) =$

nb2j0fedk9w

Example 4. Find $f(x)$.

(1) $f'(x) = 2\cos x + \sec^2 x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $f(\frac{\pi}{3}) = 4$.

(2) $f''(x) = 2e^x + 3\sin x, f(0) = 0, f(\pi) = 0$.

Solution.

□ 以上方程式稱爲「帶有初始條件的微分方程式」(ordinary differential equation)。