Chapter 3 Differentiation Rules

3.1 Derivatives of Polynomials and Exponential Functions, page 172

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Property 1 (Derivative of a constant function, page 172)**.**

$$\frac{\mathrm{d}}{\mathrm{d}x}(c) = 0.$$

Proof. Let $f(x) = c$ the constant function, then from the definition of a derivative, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$

□

Property 2 (The power rule, page 173)**.** *If n is any real number, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}.$$

Proof. Let $f(x) = x^n$. Here we check the case $n \in \mathbb{Z}$ and show the general case in Section 3.6. First, for $n \in \mathbb{N}$, by the Binomial Theorem, we compute

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{\sum_{k=0}^{n} C_k^n x^{(n-k)} h^k - x^n}{h} \\ &= \lim_{h\to 0} \sum_{k=1}^{n} C_k^n x^{(n-k)} h^{k-1} = \sum_{k=1}^{n} \left(\lim_{h\to 0} C_k^n x^{(n-k)} h^{k-1} \right) = C_1^n x^{n-1} = nx^{n-1}. \end{aligned}$$

Next, we check the case negative integer $-n, n \in \mathbb{N}$. That is, let $f(x) = x^{-n}$, and we will prove $f'(x) = -nx^{-n-1}$:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1}{h} \left(\frac{1}{(x+h)^n} - \frac{1}{x^n} \right) \\ &= \lim_{h\to 0} - \frac{\sum\limits_{k=0}^{n} C_k^n x^{(n-k)} h^k - x^n}{h(x+h)^n x^n} = \lim_{h\to 0} - \sum_{k=1}^{n} \frac{C_k^n x^{(n-k)} h^{k-1}}{(x+h)^n x^n} \\ &= -\sum_{k=1}^{n} \left(\lim_{h\to 0} \frac{C_k^n x^{(n-k)} h^{k-1}}{(x+h)^n x^n} \right) = -C_1^n x^{-n-1} = -nx^{-n-1}. \end{aligned}$$

□

☐ n 次方差公式: $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$。

☐ 活用公式: $a - b = (a^{\frac{1}{n}} - b^{\frac{1}{n}})(a^{\frac{n-1}{n}} + a^{\frac{n-2}{n}} b^{\frac{1}{n}} + \cdots + a^{\frac{1}{n}} b^{\frac{n-2}{n}} + b^{\frac{n-1}{n}})$。

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Property 3 (The constant multiple rule, page 175)**.** *If c is a constant and f is a differential function, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}(cf(x)) = c\frac{\mathrm{d}}{\mathrm{d}x}f(x).$$

Proof. Let $g(x) = cf(x)$. Then

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = \lim_{h\to 0} c\left(\frac{f(x+h)-f(x)}{h}\right) \\ &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = cf'(x). \end{aligned}$$

□

Property 4 (The sum and difference rule, page 176)**.** *If f and g are both differentiable, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)\pm g(x)) = \frac{\mathrm{d}}{\mathrm{d}x}f(x) \pm \frac{\mathrm{d}}{\mathrm{d}x}g(x).$$

Proof. Let $F(x) = f(x) \pm g(x)$. Then

$$\begin{aligned} F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{(f(x+h)\pm g(x+h)) - (f(x)\pm g(x))}{h} \\ &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h} \pm \frac{g(x+h)-g(x)}{h}\right) \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \pm \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f'(x)\pm g'(x). \end{aligned}$$

□

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Example 5. Compute the derivative of the exponential function $f(x) = a^x$.

Solution.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = \lim_{h\to 0}\frac{a^x a^h - a^x}{h} \\ &= \lim_{h\to 0}\frac{a^x(a^h-1)}{h} = \left(\lim_{h\to 0}\frac{a^h-1}{h}\right)a^x = f'(0)f(x). \end{aligned}$$

Definition 6 (the number e, page 180)**.** e is the number such that

$$\lim_{h\to 0}\frac{\mathrm{e}^h-1}{h} = 1.$$

□ $\mathrm{e} \approx 2.71828\ldots$ (回想 Section 1.4 的介紹, 到 Section 3.6 將證明 $\lim_{n\to\infty}(1+\frac{1}{n})^n = \mathrm{e}$)。

Property 7 (Derivative of the natural exponential function, page 178)**.**

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x) = \mathrm{e}^x.$$

Proof. Let $f(x) = \mathrm{e}^x$, then

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\mathrm{e}^x(\mathrm{e}^h-1)}{h} = \left(\lim_{h\to 0}\frac{\mathrm{e}^h-1}{h}\right)\mathrm{e}^x = \mathrm{e}^x.$$

□

3.2 The Product and Quotient Rules, page 183

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Property 1 (The product rule, page 184)**.** *If f and g are both differentiable, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) = g(x)\frac{\mathrm{d}}{\mathrm{d}x}f(x) + f(x)\frac{\mathrm{d}}{\mathrm{d}x}g(x).$$

Proof. Let $F(x) = f(x)g(x)$, then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)g(x+h)+f(x)\left(\frac{g(x+h)-g(x)}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)\lim_{h\to 0}g(x+h)+f(x)\lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right)\\
&= f'(x)g(x)+f(x)g'(x).
\end{aligned}$$

□

☐ 注意乘法法則 $(f(x)g(x))' \neq f'(x)g'(x)$。

☐ 推廣: $(f(x)g(x)h(x))' = \underline{f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x)}$.

☐ 萊布尼茲法則 (Leibniz Rule): $(fg)^{(n)}(x) = \sum_{k=0}^{n} C_k^n f^{(n-k)}(x)g^{(k)}(x)$.

Property 2 (The quotient rule, page 186)**.** *If f and g are both differentiable, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{\mathrm{d}}{\mathrm{d}x}f(x)-f(x)\frac{\mathrm{d}}{\mathrm{d}x}g(x)}{(g(x))^2}.$$

Proof. Let $F(x) = \frac{f(x)}{g(x)}$. Then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{f(x+h)g(x)-g(x+h)f(x)}{hg(x+h)g(x)}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-g(x+h)f(x)}{hg(x+h)g(x)}\\
&= \lim_{h\to 0}\frac{(f(x+h)-f(x))g(x)-f(x)(g(x+h)-g(x))}{hg(x+h)g(x)}\\
&= \lim_{h\to 0}\frac{\frac{f(x+h)-f(x)}{h}g(x)-f(x)\frac{g(x+h)-g(x)}{h}}{g(x+h)g(x)}\\
&= \frac{\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}g(x)-f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}}{\lim_{h\to 0}g(x+h)g(x)} = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}.
\end{aligned}$$

□

☐ 分子的兩項係數, 一正一負, 如何確定何者爲正何者爲負?

HfjoNHEoqqA

Example 3. Compute $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{f(x)}{g(x)}\right)$.

Solution. We compute

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{f(x)}{g(x)}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{gf'-fg'}{g^2}\right) \\
&= \frac{g^2(gf'-fg')'-(gf'-fg')(g\cdot g)'}{g^4} \\
&= \frac{g^2(g'f'+gf''-f'g'-fg'')-(gf'-fg')(2g'g)}{g^4} \\
&= \frac{g(gf''-fg'')-2g'(gf'-fg')}{g^3}.
\end{aligned}$$

☐ 函數 $\frac{f(x)}{g(x)}$ 微分 n 次, 分母一定可以化簡成 $(g(x))^{n+1}$。

Example 4. The curve $y = \frac{1}{1+x^2}$ is called a *witch of Maria Agnesi* (箕舌線). Find an equation of the tangent line to this curve at the point $(-1, \frac{1}{2})$.

Solution. Since

$$y'(x) = \frac{(1+x^2)(1)'-1(1+x^2)'}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}, \quad y'(-1) = \frac{(-2)\cdot(-1)}{(1+(-1)^2)^2} = \frac{1}{2}.$$

we know the tangent line equation is $y - \frac{1}{2} = \frac{1}{2}(x+1)$.

3.3 Derivatives of Trigonometric Functions, page 190

Goal: Find the derivative of six trigonometric functions.

Example 1 (page 192–193)**.** Calculate the derivative of $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$.

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Solution. We compute

$$\begin{aligned} f'(\theta) &= \lim_{h\to 0}\frac{f(\theta+h)-f(\theta)}{h} = \lim_{h\to 0}\frac{\sin(\theta+h)-\sin\theta}{h} = \lim_{h\to 0}\frac{2\cos\left(\theta+\frac{h}{2}\right)\sin\frac{h}{2}}{h} \\ &= \lim_{h\to 0}\cos\left(\theta+\frac{h}{2}\right)\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}} = \cos\theta \\ g'(\theta) &= \lim_{h\to 0}\frac{g(\theta+h)-g(\theta)}{h} = \lim_{h\to 0}\frac{\cos(\theta+h)-\cos\theta}{h} = \lim_{h\to 0}\frac{-2\sin\left(\theta+\frac{h}{2}\right)\sin\frac{h}{2}}{h} \\ &= \lim_{h\to 0}-\sin\left(\theta+\frac{h}{2}\right)\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}} = -\sin\theta. \end{aligned}$$

Example 2 (page 193)**.** Calculate the derivative of $\tan\theta, \cot\theta, \sec\theta$, and $\csc\theta$.

Solution. Using the quotient rule, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta}\tan\theta &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\sin\theta}{\cos\theta}\right) = \frac{\cos\theta(\sin\theta)'-\sin\theta(\cos\theta)'}{\cos^2\theta} = \frac{\cos\theta\cos\theta-\sin\theta(-\sin\theta)}{\cos^2\theta} \\ &= \frac{1}{\cos^2\theta} = \sec^2\theta \\ \frac{\mathrm{d}}{\mathrm{d}\theta}\cot\theta &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{\tan\theta}\right) = \frac{(\tan\theta)(1)'-1(\tan\theta)'}{\tan^2\theta} = -\frac{\sec^2\theta}{\tan^2\theta} = -\csc^2\theta \\ \frac{\mathrm{d}}{\mathrm{d}\theta}\sec\theta &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{\cos\theta}\right) = \frac{(\cos\theta)(1)'-1(\cos\theta)'}{\cos^2\theta} = \frac{\sin\theta}{\cos^2\theta} = \sec\theta\tan\theta \\ \frac{\mathrm{d}}{\mathrm{d}\theta}\csc\theta &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{\sin\theta}\right) = \frac{(\sin\theta)(1)'-1(\sin\theta)'}{\sin^2\theta} = -\frac{\cos\theta}{\sin^2\theta} = -\csc\theta\cot\theta. \end{aligned}$$

☐ $\tan\theta\cot\theta = 1 \Rightarrow \sec^2\theta\cot\theta + \tan\theta(\cot\theta)' = 0 \Rightarrow (\cot\theta)' = -\frac{1}{\sin^2\theta} = -\csc^2\theta$.

☐ 務必熟記六個三角函數的導函數。

Example 3. Calculate the derivative of $\sin^2\theta$ and $\cos^2\theta$.

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Solution. We compute

$$(\sin^2\theta)' = (\sin\theta\sin\theta)' = \cos\theta\sin\theta + \sin\theta\cos\theta = 2\sin\theta\cos\theta.$$

$$(\cos^2\theta)' = (\cos\theta\cos\theta)' = -\sin\theta\cos\theta - \cos\theta\sin\theta = -2\sin\theta\cos\theta.$$

☐ $\sin^2\theta + \cos^2\theta = 1 \Rightarrow (\sin^2\theta + \cos^2\theta)' = 0$.

Example 4. Compute $\frac{\mathrm{d}^n}{\mathrm{d}\theta^n}\sin\theta$ and $\frac{\mathrm{d}^n}{\mathrm{d}\theta^n}\cos\theta$.

Solution. Direct computation gives

$$\begin{aligned} f(\theta) &= \sin\theta = f^{(4)}(\theta) = f^{(4k)}(\theta) & g(\theta) &= \cos\theta = g^{(4)}(\theta) = g^{(4k)}(\theta) \\ f'(\theta) &= \cos\theta = f^{(5)}(\theta) = f^{(4k+1)}(\theta) & g'(\theta) &= -\sin\theta = g^{(5)}(\theta) = g^{(4k+1)}(\theta) \\ f''(\theta) &= -\sin\theta = f^{(6)}(\theta) = f^{(4k+2)}(\theta) & g''(\theta) &= -\cos\theta = g^{(6)}(\theta) = g^{(4k+2)}(\theta) \\ f'''(\theta) &= -\cos\theta = f^{(7)}(\theta) = f^{(4k+3)}(\theta) & g'''(\theta) &= \sin\theta = g^{(7)}(\theta) = g^{(4k+3)}(\theta). \end{aligned}$$

Remark that we have another type of formula:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta = \sin\left(\theta+\frac{\pi}{2}\right) &\Rightarrow \frac{\mathrm{d}^n}{\mathrm{d}\theta^n}\sin\theta = \sin\left(\theta+\frac{n\pi}{2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}\theta}\cos\theta = \cos\left(\theta+\frac{\pi}{2}\right) &\Rightarrow \frac{\mathrm{d}^n}{\mathrm{d}\theta^n}\cos\theta = \cos\left(\theta+\frac{n\pi}{2}\right). \end{aligned}$$

以下求導法則務必熟記

- $(c)' = 0.$ — Derivative of a constant function
- $(x^n)' = nx^{n-1}.$ — The powe rule
- $(cf(x))' = cf'(x).$ — The constant multiple rule
- $(f(x)+g(x))' = f'(x)+g'(x).$ — The sum rule
- $(f(x)-g(x))' = f'(x)-g'(x).$ — The difference rule
- $(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$ — The produce rule
- $\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}.$ — The quotient rule

⋆ $(f(g(x)))' = f'(g(x))\cdot g'(x).$ — The chain rule

三角函數的導函數, 務必熟記

$(\sin\theta)' = \cos\theta$	$(\tan\theta)' = \sec^2\theta$	$(\sec\theta)' = \sec\theta\tan\theta$
$(\cos\theta)' = -\sin\theta$	$(\cot\theta)' = -\csc^2\theta$	$(\csc\theta)' = -\csc\theta\cot\theta$

e 的定義與 e^x 的導函數

$$\lim_{h\to 0}\frac{\mathrm{e}^h-1}{h} = 1, \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x) = \mathrm{e}^x.$$

3.4 The Chain Rule, page 197

Theorem 1 (The Chain Rule (鏈鎖律), page 198)**.** *If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product*

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$$F'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}.$$

Theorem 2 (The power rule combined with the chain rule, page 200)**.** *If n is any real number and $g(x)$ is differentiable, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}(g(x))^n = n(g(x))^{n-1} \cdot g'(x).$$

Proof. The relation is $x \overset{g}{\to} u = g(x) \overset{f}{\to} y = u^n$. By the Chain Rule, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left.\frac{\mathrm{d}y}{\mathrm{d}u}\right|_{u=g(x)} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = nu^{n-1}\big|_{u=g(x)} \cdot g'(x) = n(g(x))^{n-1} \cdot g'(x).$$

□

☐ 合成函數的求導法則。

☐ 熟悉合成函數的先後順序。

☐ 洋蔥: 如果你願意一層一層一層的剝開我的心 ......

Example 3. Find the derivatives.

fc_uQKnbGEM

(a) $(\sin(ax))' =$

(b) $(\sin(x^2))' =$

(c) $(\sin^2 x)' =$

(d) $(\mathrm{e}^{2x})' =$

(e) $(\mathrm{e}^{x^2})' =$

Example 4 (page 202)**.** Show that $\frac{\mathrm{d}}{\mathrm{d}x} a^x = a^x \ln a$.

jZcVk7bCXwo

Solution.

Example 5. Let $f(x) = \frac{x}{\sqrt[3]{3x+5}}$. Find $f'(1)$.

Solution.

☐ 熟悉除法法則、鏈鎖率、代值。

Example 6. Let $y = x^{a^b} + a^{x^b} + a^{b^x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution.

☐ 想清楚這三個函數的意義與合成函數的先後順序; 清楚何時使用多項式或指數微分。

1loKxjGqQSA

Example 7. Let $f(x), g(x) \in C^1(\mathbb{R})$ and $f(1) = 1, f'(1) = 0, g(0) = 0, g'(0) = 1$. Find the limit

$$\lim_{x \to \frac{\pi}{2}} \frac{f(\sin x) g(\cos x)}{x^2 - \frac{\pi}{2} x}.$$

Solution.

☐ 將極限問題與某個函數的導數的定義連接起來。

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Example 8. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

(a) Find $f'(0)$.

(b) When $x \neq 0$, find $f'(x)$.

(c) Does $f''(0)$ exist? If it exists, please find its value. If not, give reason to support your argument.

Solution.

☐ (a) 遇到分段定義的函數, 其導數利用「定義」處理。

☐ (b) 導函數的四則運算與鏈鎖率是建立在函數都「很好」才能使用。

☐ (c) 高階導數, 遇到分段定義的函數, 仍然是由「定義」出發。

☐ 比較 Section 2.7 的 **Example 6**。

To prove the chain rule, one idea is the following:

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\ &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(g(x)) \cdot g'(x). \end{aligned}$$

This argument looks great, but it is *not* correct. What is the problem? How do we overcome the problem?

Appendix

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Proof of the Chain Rule. The function $g(x)$ is differentiable at x. This means $g'(x)$ exists and

$$\frac{g(x+h)-g(x)}{h} - g'(x) \to 0 \quad \text{as} \quad h \to 0.$$

Define a new variable v by

$$v = \frac{g(x+h)-g(x)}{h} - g'(x) \Rightarrow g(x+h) = g(x) + (g'(x)+v)h. \tag{1}$$

Notice that v depends on the number h and that $v \to 0$ as $h \to 0$. Similarly, because the function f is differentiable at the point $y = g(x)$, we have

$$\frac{f(y+k)-f(y)}{k} - f'(y) \to 0 \quad \text{as} \quad k \to 0.$$

Define another variable w by

$$w = \frac{f(y+k)-f(y)}{k} - f'(y) \Rightarrow f(y+k) = f(y) + (f'(y)+w)k. \tag{2}$$

Notice that w depends on the number k and that $w \to 0$ as $k \to 0$.

From (1), we get

$$f(g(x+h)) = f(g(x) + (g'(x)+v)h).$$

Use (2) applied to the right-hand-side with $k = (g'(x)+v)h$ and $y = g(x)$ to get.

$$f(g(x) + (g'(x)+v)h) = f(g(x)) + (f'(g(x))+w)(g'(x)+v)h.$$

Note that $k \to 0$ as $h \to 0$, and so $w \to 0$ as $h \to 0$. So

$$\begin{aligned}\frac{f(g(x+h))-f(g(x))}{h} &= \frac{f(g(x)) + (f'(g(x))+w)(g'(x)+v)h - f(g(x))}{h} \\ &= \frac{(f'(g(x))+w)(g'(x)+v)h}{h} = (f'(g(x))+w)(g'(x)+v).\end{aligned}$$

Hence

$$\lim_{h\to 0} \frac{f(g(x+h))-f(g(x))}{h} = \lim_{h\to 0}(f'(g(x))+w)(g'(x)+v)$$
$$= \left(\lim_{h\to 0} f'(g(x)) + \lim_{h\to 0} w\right)\left(\lim_{h\to 0} g'(x) + \lim_{h\to 0} v\right) = f'(g(x))g'(x).$$

since $v \to 0$ as $h \to 0$ and $w \to 0$ as $h \to 0$. □

3.5 Implicit Differentiation, page 208

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable $y = f(x)$. However, there are a lot of functions are defined implicitly by a relation x and y and we formally write it as $F(x, y) = 0$. For example, $F(x, y) = x^2 + y^2 - 4 = 0$, $F(x, y) = x^3 + y^3 - 6xy = 0$.

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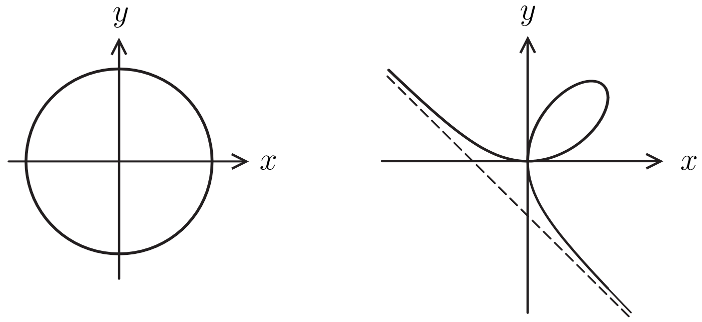

Figure 1: (a) A circle $x^2 + y^2 = 4$. (b) The folium of Descartes $x^3 + y^3 - 6xy = 0$.

Most of time, implicit functions are not "functions" (see the definition of a function in Section 1.1), but they are locally be expressed as functions.

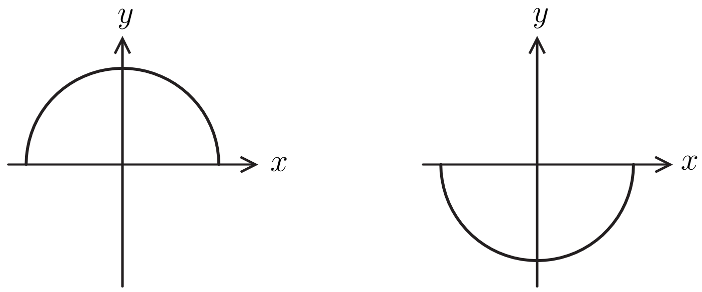

Figure 2: A circle $x^2 + y^2 = 4$.

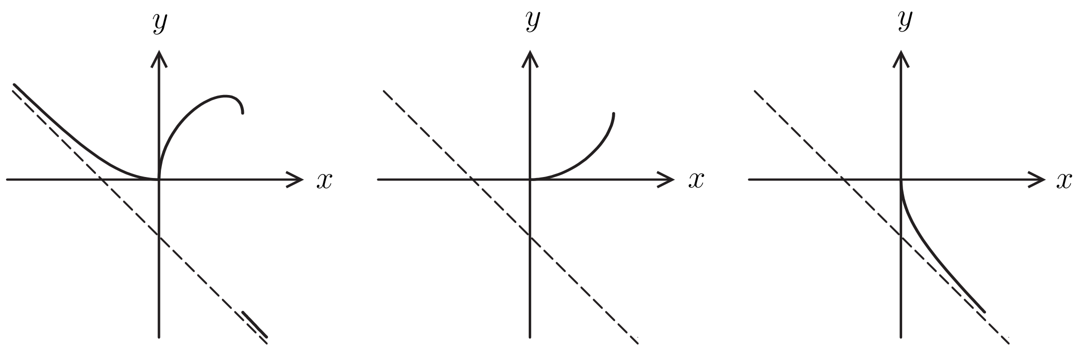

Figure 3: The folium of Descartes $x^3 + y^3 - 6xy = 0$.

Furthermore, it's not easy to solve implicit functions $F(x, y) = 0$ to explicit ones $y = f(x)$. Fortunately, we can compute the derivative of implicit functions without solving implicit functions to explicit ones by ________________.

L1UA-nYD_A4

Example 1. If $x^2 + y^2 = r^2$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution.

Solution 2.

☐ 將隱函數 $F(x, y) = 0$ 視爲 $F(x, y(x)) = 0$。

Example 2.

(a) Find y' if $x^3 + y^3 = 6xy$.

(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

(c) At what point in the first quadrant is the tangent line horizontal?

Solution.

If we solve the equation $x^3+y^3 = 6xy$ for y in terms of x, we get three functions determined by the equation:

$$y = f(x) = \sqrt[3]{-\frac{1}{2}x^3 + \sqrt{\frac{1}{4}x^6 - 8x^3}} + \sqrt[3]{-\frac{1}{2}x^3 - \sqrt{\frac{1}{4}x^6 - 8x^3}}$$

and

$$y = \frac{1}{2}\left(-f(x) \pm \sqrt{-3}\left(\sqrt[3]{-\frac{1}{2}x^3 + \sqrt{\frac{1}{4}x^6 - 8x^3}} - \sqrt[3]{-\frac{1}{2}x^3 - \sqrt{\frac{1}{4}x^6 - 8x^3}}\right)\right).$$

It is very complicated to get the derivative by these formulae.

Implicit differentiation works for a lot of equations such as $y^5 + 3x^2y^2 + 5x^4 = 12$ for which it is *impossible* to find an expression for y in terms of x.

An application of implicit differentiation is derivatives of inverse functions.

ru7QOQ54NrM

Derivatives of Inverse Trigonometric Functions (page 214)**.**

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}x=-\frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}x=\frac{1}{1+x^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cot^{-1}x=-\frac{1}{1+x^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sec^{-1}x=\frac{1}{x\sqrt{x^2-1}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\csc^{-1}x=-\frac{1}{x\sqrt{x^2-1}}.$$

Proof. Let $y=y(x)=\sin^{-1}x$, then $\sin y=x \Rightarrow \cos y\,\frac{\mathrm{d}y}{\mathrm{d}x}=1$. So

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}.$$

□

☐ $\sin^{-1}x, \cos^{-1}x, \tan^{-1}x$ 的導函數要熟記; 六個反三角函數的導函數也要會推導。

Example 3 (Derivatives of inverse functions)**.** Suppose f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Show that

$$(f^{-1})'(x)=\frac{1}{f'(f^{-1}(x))}$$

provide that the denominator is not 0.

Solution.

Example 4. Find the tangent line of $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ at (x_0,y_0) and the length between x-intercept and y-intercept.

LgLBwdDoWiI

Solution.

oAO8sqVKgK4

Example 5. Suppose that $f(x) \in C^2(\mathbb{R})$ and $f(x)$ satisfies $x^2 + xf(x) + (f(x))^2 = k$, where k is a constant, and $f'(a) = f''(a) = 1$. Find a and k.

Solution.

WVd3RqVGsuo

Example 6. Two curves are *orthogonal* (正交) if their tangent lines are perpendicular at each point of intersection. Show that the given families of curves are *orthogonal trajectories* (正交軌線) of each other; that is, every curve in one family is orthogonal to every curve in the other family.

(a) $x^2 + y^2 = r^2$, $ax + by = 0$.

(b) $x^2 + y^2 = ax$, $x^2 + y^2 = by$.

Solution.

(a)

(b) First, $x^2 + y^2 = ax \Rightarrow 2x + 2yy' = a \Rightarrow y' = \frac{a-2x}{2y}$ if $y \neq 0$. Next, $x^2 + y^2 = by \Rightarrow 2x + 2yy' = by' \Rightarrow (b-2y)y' = 2x \Rightarrow y' = \frac{2x}{b-2y}$ if $y \neq \frac{b}{2}$. So if $y \neq 0$ and $y \neq \frac{b}{2}$, we have

$$m_1 \cdot m_2 = \frac{a-2x}{2y} \cdot \frac{2x}{b-2y} = \frac{2ax - 4x^2}{2by - 4y^2} = \frac{2x^2 + 2y^2 - 4x^2}{2x^2 + 2y^2 - 4y^2} = \frac{2y^2 - 2x^2}{2x^2 - 2y^2} = -1.$$

If $y = 0$, then $x^2 - ax = x(x-a) = 0 \Rightarrow x = 0$ or $x = a$, so $x^2 + y^2 = ax$ has vertical tangent line at $x = 0$ or $x = a$. If $(x, y) = (0, 0)$, $m_2 = 0$. If $(x, y) = (a, 0), a \neq 0$, no curves in the family $x^2 + y^2 = by$ passes through $(a, 0)$. If $y = \frac{b}{2}$, then $x = \pm\frac{b}{2}$, so $x^2 + y^2 = by$ has vertical tangent line at $(\frac{b}{2}, \pm\frac{b}{2})$. At $(\frac{b}{2}, \pm\frac{b}{2})$, we get $a = b$, and $m_1 = a - 2x = b - 2\frac{b}{2} = 0$, so $x^2 + y^2 = ax$ has horizontal tangent line at $(\frac{b}{2}, \pm\frac{b}{2})$.

3.6 Derivatives of Logarithmic Function, page 218

Another application of implicit differentiation is getting the derivatives of logarithmic functions.

Example 1 (page 218)**.** Compute $\frac{\mathrm{d}}{\mathrm{d}x}(\log_a x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)$.

ze7xwBrpVZo

Solution. Let $y = \log_a x$. Then $a^y = x$. Differentiating this equation implicit with respect to x, we get

In particular, we put $a = \mathrm{e}$ then $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) =$ ___.

Example 2 (page 220)**.** Find $f'(x)$ if $f(x) = \ln|x|$.

Solution. Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0, \end{cases} \quad \text{it follows that } f'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ & \text{if } x < 0. \end{cases}$$

Thus $f'(x) =$ ___ for all $x \neq 0$.

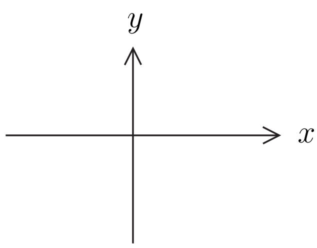

Figure 1: $f(x) = \ln|x|$.

☐ $f(x) = \ln|x|$ 的定義域是 ____________________。

Application: Logarithmic Differentiation (對數微分法)

The calculation of derivatives of complicated functions involving products, quotients, or powers can be simplified by the method of *logarithmic differentiation*.

Example 3 (page 220)**.** Differentiate $y = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$.

nTtUgqbuI0w

Solution.

pwlJ7ZGRGdE

The Power Rule (page 221)**.** *If* $n \in \mathbb{R}$ *and* $f(x) = x^n$*, then* $f'(x) = nx^{n-1}$.

Proof. Let $y = x^n$ and use logarithmic differentiation for $x \neq 0$:

If $x = 0$, by the definition of derivative, we have

$$f'(0) =$$

□

Example 4. Differentiate $y = x^x$. (The function is defined on $x > 0$)

Solution.

Solution 2.

□ 注意 x^n, a^x, x^x 變數的位置, 求導法則均不同。

The Number e as a Limit

MP5JXxzJfbo

Example 5 (page 189)**.** Show that $\mathrm{e} = \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

Solution. We have shown that if $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$. Thus $f'(1) = 1$. From the definition of a derivative as a limit and the continuity of the logarithmic function, we have

$$f'(1) = \lim_{x\to 0}\frac{f(1+x) - f(1)}{x} =$$
$$=$$

Hence we have $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \mathrm{e}$. If we put $n = \frac{1}{x}$, then $n \to \infty$ as $x \to 0^+$, then we get an alternative expression for e:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \mathrm{e}.$$

Example 6 (page 189)**.** Show that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \mathrm{e}^x$ for any $x > 0$.

Solution.

3.8 Exponential Growth and Decay, page 237

In this section, we will show some examples of quantities grow or decay at a rate proportional to their size:

xB1bb_bh_Qo

(1) The number of individuals in a population of animals or bacteria.

(2) In nuclear physics, the mass of a radioactive substance decays at a rate proportional to the mass.

(3) In chemistry, the rate of unimolecular first-order reaction is proportional to the concentration of the substance.

(4) In finance, the value of a savings account with continuously compounded interest increases at a rate proportional to that value.

Definition 1 (page 237)**.** If $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky, \tag{3}$$

where k is a constant. It is called the *law of natural growth* (if $k > 0$) or the *law of natural decay* (if $k < 0$). The equation (3) is called a *differential equation* (微分方程) because it involves an unknown function y and its derivative $\frac{\mathrm{d}y}{\mathrm{d}t}$.

Theorem 2 (page 237)**.** *The only solutions of the differential equation* $\frac{\mathrm{d}y}{\mathrm{d}t} = ky$ *are the exponential functions*

$$y(t) = y(0)\mathrm{e}^{kt}.$$

Proof. Here we check any exponential function of the form $y(t) = C\mathrm{e}^{kt}$, where C is a constant, satisfies

$$y'(t) = C(k\mathrm{e}^{kt}) = k(C\mathrm{e}^{kt}) = ky(t).$$

We will prove in section 9.4 that *any* function that satisfies $\frac{\mathrm{d}y}{\mathrm{d}t} = ky$ must be of the form $y = C\mathrm{e}^{kt}$.

To see the significance of the constant C, we observe that

$$y(0) = C\mathrm{e}^{k\cdot 0} = C.$$

Therefore C is the initial value of the function. □

☐ 唯有指數函數滿足微分方程 $\frac{\mathrm{d}y}{\mathrm{d}t} = ky$。

☐ 其他爲底數的指數函數用換底公式都可以改成以 e 爲底。

Population Growth, page 237

nrKI3anQDhg

In the context of population growth, where $P(t)$ is the size of a population at time t, we can write

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP \quad \text{or} \quad \frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t} = k.$$

The quantity $\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t}$ is the growth rate divided by the population size; it is called the *relative growth rate* (相對成長率).

Instead of saying "the growth rate is proportional to population size" we could say "the relative growth rate is constant."

Example 3. The table gives the population of India, in millions, for the second half of the 20th century.

Year	1951	1961	1971	1981	1991	2001
Population	361	439	548	683	846	1029

(a) Use the exponential model and the census figure for 1951 and 1961 to predict the population in 2001. Compare with the actual population.

(b) Use the exponential model and the census figure for 1961 and 1981 to predict the population in 2001. Compare with the actual population. Then use this model to predict the population in the year 2010 and 2020.

Solution.

(a)

(b) $P(t) = P(0)\mathrm{e}^{kt} = 439\mathrm{e}^{kt}$, $P(20) = 439\mathrm{e}^{20k} = 683 \Rightarrow k = \frac{1}{20}\ln\frac{683}{439} \doteq 0.022099$.

$P(40) = 439\mathrm{e}^{40k} = 439\mathrm{e}^{0.88396} \doteq 1063$, $P(49) = 439\mathrm{e}^{49k} = 439\mathrm{e}^{1.08289} \doteq 1296$.

$P(59) = 439\mathrm{e}^{59k} = 439\mathrm{e}^{1.30389} \doteq 1617$.

Exercise 4.

(a) Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century.

(b) What is the relative growth rate?

(c) Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

Radioactive Decay, page 239

Radioactive substances decay by spontaneously emitting radiation.

xZgT0oQT2e4

If $m(t)$ is the mass remaining from an initial mass m_0 of the substance after time t, then the relative decay rate

$$-\frac{1}{m}\frac{\mathrm{d}m}{\mathrm{d}t}$$

has been found experimentally to be constant. It follows that

$$\frac{\mathrm{d}m}{\mathrm{d}t} = km,$$

where k is a negative constant. The solution is $m(t) = m_0\mathrm{e}^{kt}$.

Physicist express the rate of decay in terms of *half-life* (半衰期), the time required for half of any given quantity to decay.

Example 5. Scientists can determine the age of ancient objects by the method of *radiocarbon dating* (放射性碳紀年). The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, ^{14}C, with a half-life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates ^{14}C food chains. When a plant or animal dies, it stops replacing its carbon and the amount of ^{14}C begins to decrease through radioactive decay. Therefore the level of radioactivity must also decay exponentially.

A parchment fragment was discovered that had about 74% as much ^{14}C radioactivity as does plant material on the earth today. Estimate the age of the parchment.

Solution.

Exercise 6. Experiments show that if the chemical reaction

$$N_2O_5 \to 2NO_2 + \frac{1}{2}O_2$$

takes place at 45°C, the rate of reaction of dinitrogen pentoxide is proportional to its concentration as follows:

$$-\frac{\mathrm{d}[N_2O_5]}{\mathrm{d}t} = 0.0005[N_2O_5]$$

(a) Find an expression for the concentration $[N_2O_5]$ after t seconds if the initial concentration is C.

(b) How long will the reaction take to reduce the concentration of N_2O_5 to 90% of its original value?

Newton's Law of Cooling, page 240

JVeHbxx5698

Newton's Law of Cooling (牛頓冷卻定律) states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provides that this difference is not too large.

If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings, then we can formulate Newton's Law of Cooling as a differential equation:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_s),$$

where k is a constant.

When we make the change of variable $y(t) = T(t) - T_s$. Since T_s is constant, we have $y'(t) = T'(t)$ and the equation becomes

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky.$$

Hence we can solve y first and then find T.

☐ 牛頓冷卻定律也適用於物體增溫, 例如魚從冰箱拿至室溫解凍。

Example 7. In a murder investigation, the temperature of the corpse was 32.5°C at 1:30 PM and 30.3°C and hour later. Normal body temperature is 37.0°C and the temperature of the surroundings was 20.0°C. When did the murder take place?

Solution.

Exercise 8. A roast turkey is taken from an oven when its temperature has reached 85°C and is placed on a table in a room where the temperature is 22°C.

(a) If the temperature of the turkey is 65°C after half an hour, what is the temperature after 45 minutes?

(b) When will the turkey have cooled to 40°C?

Question 9. 兩杯熱咖啡, 一杯立馬加奶精, 然後放置五分鐘; 另一杯先放五分鐘後再加奶精, 哪一杯會比較熱? (假設奶精的溫度比室溫低, 室溫又比咖啡低。)

Continuously Compounded Interest, page 241

If an amount A_0 is invested at an interest rate r, and interest is compounded n times a year, then in each compounding period the interest rate is $\frac{r}{n}$ and there are nt compounding periods in t years, so after t years the value of the investment is

Ez4tMY1N1og

$$A_0\left(1+\frac{r}{n}\right)^{nt}.$$

The interest paid increases as the number of compounding periods n increases. If we let $n \to \infty$, then we will be compounding the interest *continuously* (連續複利) and the value of the investment will be

$$A(t) = \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt} =$$

The above equation gives

$$\frac{\mathrm{d}A}{\mathrm{d}t} = rA(t),$$

which says that, with continuous compounding of interest, the rate of increase of an investment is proportional to its size.

Example 10.

(a) How long will it take an investment to double in value if the interest rate is 6% compounded continuously?

(b) What is the equivalent annual interest rate?

Solution.

Exercise 11.

(a) If $3000 is invested at 5% interest, find the value of the investment at the end of 5 years if the interest is compounded (1) annually, (2) semiannually, (3) monthly, (4) weekly, (5) daily, and (6) continuously.

(b) If $A(t)$ is the amount of the investment at time t for the case of continuous compounding, write a differential equation and an initial condition satisfied by $A(t)$.

3.9 Related Rates, page 245

2ncz2b8oWEw

Idea: Compute the rate of increase of one quantity in terms of the rate of change of another quantity (which may be more easily measured).

Procedure:

(1) Draw a picture or a diagram if possible.

(2) Introduce notation. Assign symbols to all quantities that are functions of time.

(3) Find an equation that relates the two quantities and then use the Chain Rule to differentiate both side with respect to time.

(4) Substitute the given information into the equation and get the unknown rate.

Example 1 (page 245)**.** Air is being pumped into a spherical balloon so that its volume increases at a rate of $100\,\mathrm{cm}^3/\mathrm{s}$. How fast is the radius of the balloon increasing when the diameter is $50\,\mathrm{cm}$?

Solution.

shnV2_3c-DM

Example 2 (page 246)**.** A ladder $5\,\mathrm{m}$ long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of $1\,\mathrm{m/s}$, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is $3\,\mathrm{m}$ from the wall?

Solution.

☐ $\frac{dy}{dt}$ 的“負號” 代表梯頂離地以 $\frac{3}{4}$ m/s 之變化率“減少”。

Example 3 (page 246)**.** A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of $2\,\mathrm{m}^3/\mathrm{min}$, find the rate at which level is rising the water is 3 m deep.

zF1yc-EnoLg

Solution.

The water of level is rising at a rate of ________.

Example 4 (page 247)**.** Car A is traveling west at 90 km/h and car B is traveling north at 100 km/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 60 m and car B is 80 m from the intersection?

EW-91hJrUt8

Solution.

The cars are approaching each other at a rate of ________.

Example 5 (page 248)**.** A man walks along a straight path at a speed of 1.5 m/s. A searchlight is located on the ground 6 m from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 8 m from the point on the path closest to the searchlight?

u4cDnQPe13c

Solution.

The searchlight is rotating at a rate of ________.

3.10 Linear Approximations and Differentials, page 251

Recall two high school mathematics questions.

oqMZvouMfL8

Example 1. Let $f(x) = 3x^3 - 22x^2 + 54x - 43$. Find $f(2.001)$ correct to three decimal place.

Solution.

Example 2. Find 1.0001^{100} correct to two decimal place.

Solution.

☐ 如何抓出一個量的「主要部份」?

qoKOAYTi_-Q

A curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line. This observation is the basis for a method of finding approximate values of functions.

The idea is that it might be easy to calculate a value $f(a)$ of a function, but difficult to compute nearby values of f. So we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$.

Given a curve $y = f(x)$, an equation of the tangent line at $(a, f(a))$ is

$$y = f(a) + f'(a)(x - a).$$

Definition 3 (page 252)**.** The approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the *linear approximation* or *tangent line approximation* of f at a (線性估計, 切線估計). The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the *linearization* of f at a (線性化).

Example 4 (page 252)**.** Approximate the numbers $\sqrt{3.98}$.

0i-B1dHTOGw

Solution.

☐ 選取「適合的點」進行線性估計。

Example 5. Using a linear approximation to estimate $\cot 46°$.

Solution.

☐ 角的單位必須換成「弧度」再計算。

Example 6. Go back to **Example 1.** and **Example 2.** to find out the calculation is in fact the linear approximation.

Solution.

(1) $f(x) = 3x^3 - 22x^2 + 54x - 43 = 1 + 2(x-2) - 4(x-2)^2 + 3(x-2)^3$.

$f'(x) = 2 - 8(x-2) + 9(x-2)^2$. $f(2.001) \approx f(2) + f'(2)(2.001-2) = 1.002$.

(2) $f(x) = (1+x)^{100} = C_0^{100}1^{100}x^0 + C_1^{100}1^{99}x^1 + C_2^{100}1^{98}x^2 + \cdots + C_{100}^{100}1^0x^{100}$.

$f'(x) = 100(1+x)^{99}$. $f(0.0001) \approx f(0) + f'(0)(0.0001-0) = 1.01$.

Applications to Physics

Differentials

The ideas behind linear approximations are sometimes formulated in the terminology and notation of *differentials* (微分). If $y = f(x)$, where f is a differentiable function, then the *differential* $\mathrm{d}x$ is an independent variable; that is, $\mathrm{d}x$ can be given the value of any real number. The *differential* $\mathrm{d}y$ is then defined in terms of $\mathrm{d}x$ by the equation

azwJe2zE1-Q

$$\mathrm{d}y \overset{\text{def.}}{=} f'(x)\,\mathrm{d}x.$$

So $\mathrm{d}y$ is a dependent variable; it depends on the values of x and $\mathrm{d}x$.

The geometric meaning of differentials is shown in Figure 1.

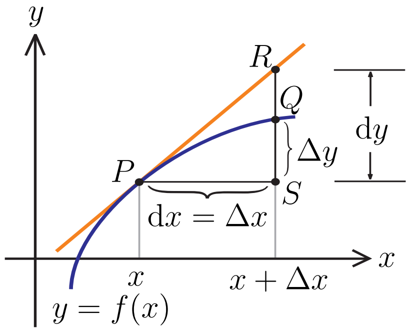

Figure 1: Geometric meaning of differentials.

Let $P(x, f(x))$ and $Q(x+\Delta x, f(x+\Delta x))$ be points on the graph of f and let $\mathrm{d}x = \Delta x$. The corresponding change in y is $\Delta y = f(x+\Delta x) - f(x)$. The slope of the tangent line PR is the derivative $f'(x)$. Thus the directed distance from S to R is $f'(x)\,\mathrm{d}x = \mathrm{d}y$. Therefore $\mathrm{d}y$ represents the amount that the tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount $\mathrm{d}x$.

hnPkx561rSo

Example 7 (page 254)**.** Compare the values of Δy and $\mathrm{d}y$ if $y = f(x) = x^3 + x^2 - 2x + 1 = 9 + 14(x-2) + 7(x-2)^2 + (x-2)^3$ and x changes from 2 to 2.05.

Solution. Since $f(2) = 9$ and $f(2.05) = 9 + 14 \cdot 0.05 + 7 \cdot (0.05)^2 + (0.05)^3 = 9.717625$, we have

☐ 了解實際誤差 Δy 與微分 $\mathrm{d}y$ 的差別。

Example 8 (page 255)**.** The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere.

Solution.

☐ 看到這樣的誤差, 你對這個球形物的「設計」感到滿意嗎?

Although the possible error in Example 8 may appear to be rather large, a better picture of the error is given by the *relative error* (相對誤差), which is computed by dividing the error by the total volume:

$$\frac{\Delta V}{V} \sim \frac{\mathrm{d}V}{V} = \frac{4\pi r^2 \mathrm{d}r}{\frac{4}{3}\pi r^3} = 3\frac{\mathrm{d}r}{r}.$$

Thus the relative error in the volume is about three times the relative error in the radius. In Example 8 the relative error in the radius is approximately

$$\frac{\mathrm{d}r}{r} = \frac{0.05}{21} \sim 0.0024$$

and it produces a relative error of about 0.007 in the volume. The error could also be expressed as *percentage error* (誤差百分比) of 0.24% in the radius and 0.7% in the volume.

3.11 Hyperbolic Functions, page 259

Hyperbolic Functions, page 259

qP-cbPmStGc

Certain combinations of e^x and e^{-x} arise so frequently in mathematics and engineering. In many ways they are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason they are called *hyperbolic functions* (雙曲函數) and individually called *hyperbolic sine, hyperbolic cosine*, and so on.

Definition 1 (Definition of the hyperbolic functions, page 259)**.**

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$
$$\coth x = \frac{\cosh x}{\sinh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x}.$$

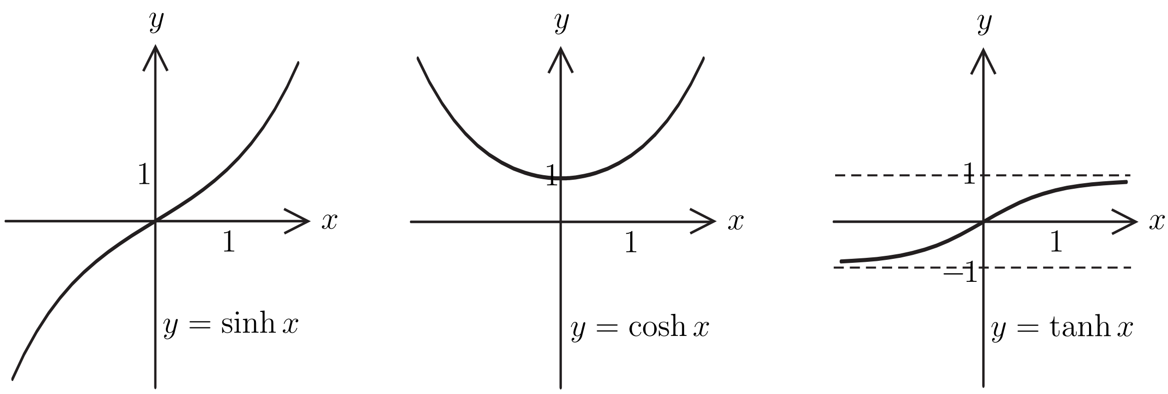

Figure 1: Hyperbolic functions.

Hyperbolic Identities (page 260)**.**

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$
$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

The identity $\cosh^2 x - \sinh^2 x = 1$ indicates the curve $(\cosh x, \sinh x)$ is hyperbola.

Derivatives of Hyperbolic Functions (page 261)**.**

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\cosh x = \sinh x$$
$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \qquad \frac{d}{dx}\coth x = -\operatorname{csch}^2 x$$
$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x \qquad \frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x$$

Inverse Hyperbolic Functions

Since the hyperbolic functions are defined in terms of exponential functions, the inverse hyperbolic functions can be expressed in terms of logarithms:

d7G00wozXe8

$$\begin{aligned}\sinh^{-1} x &= \ln\left(x + \sqrt{x^2+1}\right)\\ \cosh^{-1} x &= \ln\left(x + \sqrt{x^2-1}\right)\\ \tanh^{-1} x &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\end{aligned}$$

Derivatives of Inverse Hyperbolic Functions (page 263)**.**

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1} x &= \frac{1}{\sqrt{1+x^2}} & \frac{\mathrm{d}}{\mathrm{d}x}\cosh^{-1} x &= \frac{1}{\sqrt{x^2-1}}\\ \frac{\mathrm{d}}{\mathrm{d}x}\tanh^{-1} x &= \frac{1}{1-x^2} & \frac{\mathrm{d}}{\mathrm{d}x}\coth^{-1} x &= \frac{1}{1-x^2}\\ \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{sech}^{-1} x &= -\frac{1}{x\sqrt{1-x^2}} & \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{csch}^{-1} x &= -\frac{1}{|x|\sqrt{x^2+1}}\end{aligned}$$

Example 2 (page 265)**.** Using principles from physics it can be shown that when a cable is hung between two poles, it takes the shape of a curve $y = f(x)$ that satisfies the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\rho g}{T}\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2},$$

where ρ is the linear density of the cable, g is the acceleration due to gravity, T is the tension in the cable at its lowest point, and the coordinate system is chosen appropriately.

The function

$$y = f(x) = \frac{T}{\rho g}\cosh\left(\frac{\rho g}{T}x\right)$$

is a solution of this differential equation.

☐ A curve with equation $y = c + a\cosh\left(\frac{x}{a}\right)$ is called a *catenary* (懸鏈線).

Example 3 (page 265)**.** Another application of hyperbolic functions occurs in the description of ocean waves: The velocity of a water wave with length L moving across a body of water with depth d is modeled by the function

$$v = \sqrt{\frac{gL}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)},$$

where g is the acceleration due to gravity.