Chapter 2 Limits and Derivatives

2.1 The Tangent and Velocity Problem, page 78

Question 1. What will we learn in the Calculus course?

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The tangent problem, page 78

Example 2. Plot the parabola $f(x) = x^2$. Observe all secant lines (割線) passing through the point $P(1, f(1))$ and $Q_{\Delta x}(1 + \Delta x, f(1 + \Delta x))$, where $\Delta x \neq 0$ is a number close to 0.

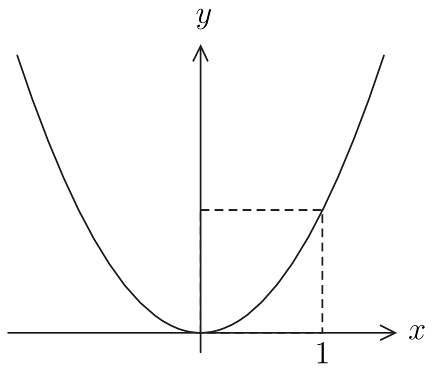

Figure 1: The parabola $f(x) = x^2$ and secant lines passing through $P(1, 1)$.

Solution. We can compute the slope of secant line $L_{PQ_{\Delta x}}$ to get

$$m_{PQ_{\Delta x}} =$$

$$=$$

So the equation of secant line $L_{PQ_{\Delta x}}$ is ________________________. When Δx is close to 0, the slope $m_{PQ_{\Delta x}}$ is close to 2. That means the family of secant lines $L_{PQ_{\Delta x}}$ is close to the line $y - 1 = 2(x - 1)$, which passes through $P(1, f(1))$ and the slope is 2.

We call $y - 1 = 2(x - 1)$ the *tangent line* (切線) of $f(x) = x^2$ at $x = 1$.

The velocity problem, page 80

WeT8xTC5zS8

- ☐ 汽車與機車的儀表板或自行車的碼表, 記錄里程並顯示“瞬時速度”。
- ☐ 棒球投手投球瞬間的速度; 網球及羽球比賽球員揮拍或殺球的球速 (大螢幕顯示的數值)。

Example 3. Suppose that a ball is dropped from the upper observation deck of Taipei 101. Find the velocity of the ball after 5 seconds.

Solution. If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation

$$s(t) = \frac{1}{2} \cdot 9.8 \cdot t^2 = 4.9t^2.$$

We can approximate the velocity at instant time $t = 5$ by computing the average velocity over the brief time interval

$$\begin{aligned}\text{average velocity} &= \frac{\text{change in position}}{\text{time elapsed}} = \frac{s(5+10^{-n}) - s(5)}{(5+10^{-n}) - 5} \\ &= \frac{4.9 \cdot ((5+10^{-n})^2 - 5^2)}{10^{-n}} = \frac{4.9 \cdot (5+10^{-n}+5)(5+10^{-n}-5)}{10^{-n}} \\ &= 4.9 \cdot (10 + 10^{-n}) = 49 + 4.9 \cdot 10^{-n}.\end{aligned}$$

That is,

Time interval	Average velocity (m/s)
$5 \le t \le 5.1$	49.49
$5 \le t \le 5.01$	49.049
$5 \le t \le 5.001$	49.0049
$5 \le t \le 5.0001$	49.00049
$5 \le t \le 5.00001$	49.000049

It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The *instantaneous velocity* (瞬時速度) when $t = 5$ is defined to be the limiting value of these average velocities over shorter an shorter time periods that start at $t = 5$. Thus the instantaneous velocity after 5 second is $v = 49$ m/s.

Remark 4. Time periods 10^{-n} we choose in **Example 3** are just some samples. In general, we can use Δt to represent any time interval and do the same calculation to get the average velocity form 5 to $5 + \Delta t$ is $4.9 \cdot (10 + \Delta t)$. The average velocity is becoming closer to 49 m/s as well when we shorten the time period.

2.2 The Limit of a Function, page 83

(One sided) Limit

qI6ZLkjJq1U

Definition 1 (page 88)**.** We write

$$\lim_{x \to a^-} f(x) = L$$

and say the *left-hand limit of* $f(x)$ *as* x *approaches* a (or the *limit of* $f(x)$ *as* x *approaches* a *from the left*) (左極限) is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

Similarly, if we require that x be greater than a, we get "the *right-hand limit of* $f(x)$ *as* x *approaches* a (右極限) is equal to L" and we write $\lim_{x \to a^+} f(x) = L$.

☐ 記號 "$x \to a^-$" 代表只考慮 $x < a$ 的部分; 而 "$x \to a^+$" 只考慮 $x > a$ 的部分。

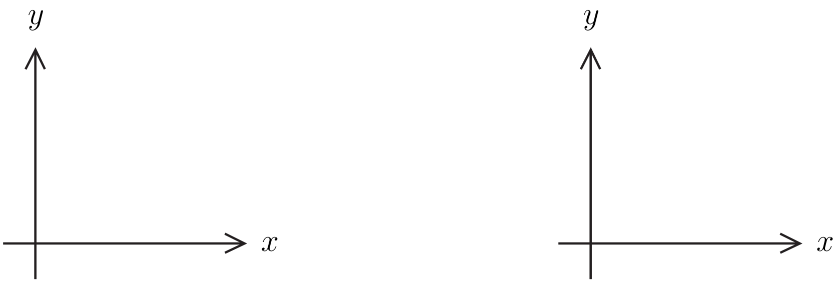

Figure 1: Left-hand limit and right-hand limit.

Definition 2 (The limit of a function, page 83)**.** Suppose $f(x)$ is defined when x is near the number a. Then we write $\lim_{x \to a} f(x) = L$ if we can make the value of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a but not equal to a.

☐ 極限 "$\lim_{x \to a} f(x) = L$" 有時候會記做 "$f(x) \to L$ as $x \to a$"。

☐ 考慮極限 $\lim_{x \to a} f(x)$ 時, 函數值 $f(a)$「不重要」。

☐ 極限 $\lim_{x \to a} f(x)$ 是在研究 $x = a$「附近」的行爲。

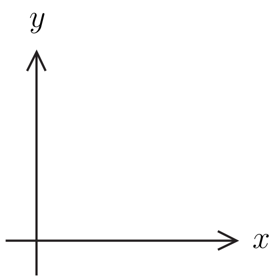

Figure 2: Limit of a function.

☐ $\lim_{x \to a} f(x) = L$ 若且唯若 (if and only if) $\lim_{x \to a^-} f(x) = L$ 且 $\lim_{x \to a^+} f(x) = L$。

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Example 3. Find the limit of the Heaviside function at $x = 0$.

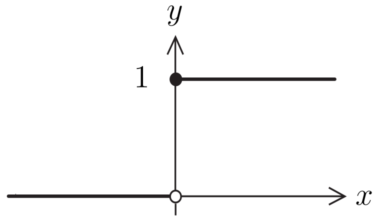

Figure 3: The Heaviside function $H(x)$.

Solution. $\lim_{x\to 0^-} H(x) =$ ____; $\lim_{x\to 0^+} H(x) =$ ____; $\lim_{x\to 0} H(x)$ __________.

Example 4. The graph of a function $f(x)$ is shown in Figure 4. Use it to state the values (if they exist) of the following:

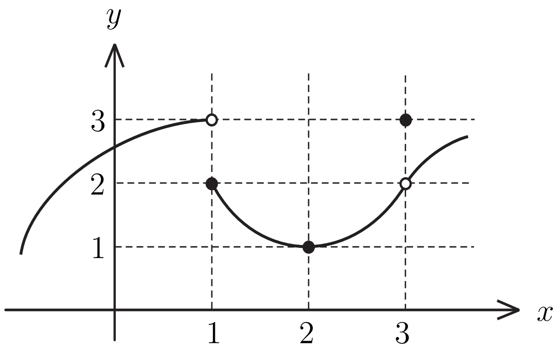

Figure 4: The graph of $f(x)$.

(a1) $\lim_{x\to 1^-} f(x)$	(b1) $\lim_{x\to 1^+} f(x)$	(c1) $\lim_{x\to 1} f(x)$	(d1) $f(1)$
(a2) $\lim_{x\to 2^-} f(x)$	(b2) $\lim_{x\to 2^+} f(x)$	(c2) $\lim_{x\to 2} f(x)$	(d2) $f(2)$
(a3) $\lim_{x\to 3^-} f(x)$	(b3) $\lim_{x\to 3^+} f(x)$	(c3) $\lim_{x\to 3} f(x)$	(d3) $f(3)$

Example 5. Observe the function $f(x) = \frac{\sin x}{x}$ and guess the value of $\lim_{x\to 0} \frac{\sin x}{x}$.

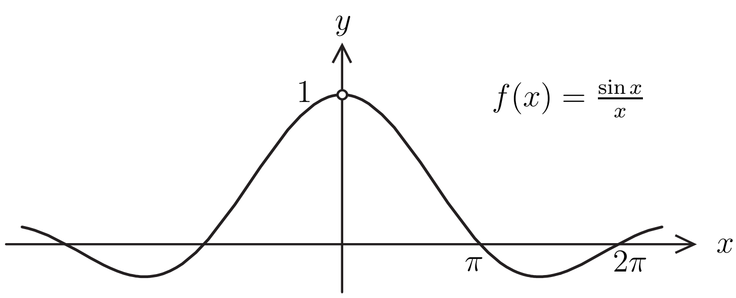

Figure 5: The graph of $f(x) = \frac{\sin x}{x}$.

Example 6. Guess the limit $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ and $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$.

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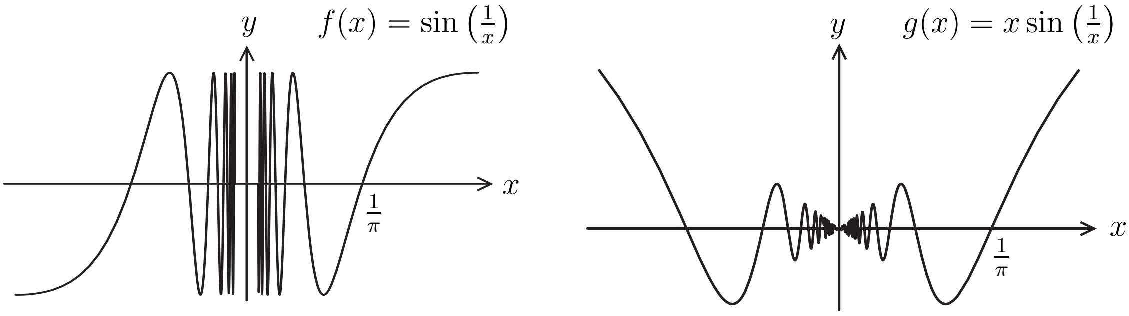

Figure 6: The graph of $f(x) = \sin\left(\frac{1}{x}\right)$ and $g(x) = x\sin\left(\frac{1}{x}\right)$.

☐ 這個例題遇到了什麼困難?

Infinite Limits

Definition 7 (page 89)**.** Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

Definition 8 (page 94)**.** Let f be a function defined on both sides of a, except possibly at a itself. Then $\lim_{x\to a} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily negative by taking x sufficiently close to a, but not equal to a.

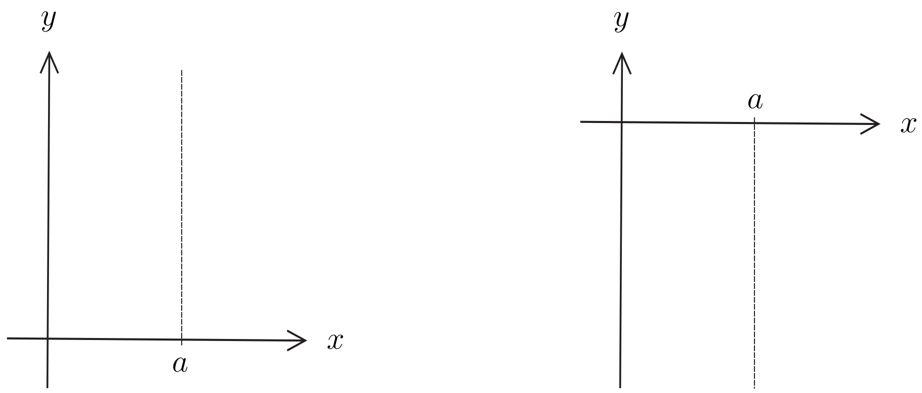

Figure 7: Infinite limit $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} f(x) = -\infty$.

☐ 極限 $\lim_{x\to a} f(x) = \infty$ 也可以寫成 "$f(x) \to \infty$ as $x \to a$."

Similar definition can be given for the one-sided infinite limits:

$$\lim_{x\to a^-} f(x) = \infty \quad \lim_{x\to a^+} f(x) = \infty \quad \lim_{x\to a^-} f(x) = -\infty \quad \lim_{x\to a^+} f(x) = -\infty.$$

Definition 9 (page 90)**.** The line $x = a$ is called a *vertical asymptote* (垂直漸近線) of the curve $y = f(x)$ if at least one of the following statement is true:

$$\begin{array}{lll} \lim_{x\to a} f(x) = \infty & \lim_{x\to a^-} f(x) = \infty & \lim_{x\to a^+} f(x) = \infty \\ \lim_{x\to a} f(x) = -\infty & \lim_{x\to a^-} f(x) = -\infty & \lim_{x\to a^+} f(x) = -\infty. \end{array}$$

Example 10.

(a) $f(x) = \tan x$ has vertical asymptotes ____________________.

(b) $f(x) = \sec x$ has vertical asymptotes ____________________.

(c) $f(x) = \frac{1}{x}$ has a vertical asymptote ______________.

(d) $f(x) = \ln x$ has a vertical asymptote ______________.

2.3 Calculating Limits Using the Limit Laws, page 95

MBNlhcOSmVY

Theorem 1. *If* $\lim_{x\to a} f(x)$ *exists, then it is unique.*

Theorem 2 (Limit laws, page 95)**.** *Suppose that c is a constant and the limits* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist. Then*

(1) $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$ (Sum Law)

(2) $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x).$ (Difference Law)

(3) $\lim_{x\to a}(cf(x)) = c\lim_{x\to a} f(x).$ (Constant Multiple Law)

(4) $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x).$ (Product Law)

(5) $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} g(x) \neq 0.$ (Quotient Law)

The followings are some special limits:

(6) $\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$ *where* $n\in\mathbb{N}.$ (Power Law)

(7) $\lim_{x\to a} c = c.$

(8) $\lim_{x\to a} x = a.$

(9) $\lim_{x\to a} x^n = a^n$ *where* $n\in\mathbb{N}.$

(10) $\lim_{x\to a}\sqrt[n]{x} = \sqrt[n]{a}$ *where* $n\in\mathbb{N}$. (*If n is even, we assume* $a>0$.)

(11) $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ *where* $n\in\mathbb{N}$. (*If n is even, we assume* $\lim_{x\to a} f(x) > 0$.)

☐ 使用定理前, 必須檢查“所有條件”是否均成立。

☐ 上述定理亦適用於單側極限。

☐ 極限的四則運算可推廣至「有限個」的操作 (數學歸納法)。

Example 3. Find the limit $\lim_{x\to 1}\frac{x^2-1}{x-1}$. (比較 Section 2.1, **Example 1**.)

jbykJ6niFnc

Solution. Let $x = 1+\Delta x$, then $x\to 1$ is equivalent to $\Delta x\to 0$, and

Solution 2.

☐ 解法一, 利用變數變換。解法二, 要先將函數適當整理後, 再用極限定理。

Example 4. Find the limit $\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}$.

Solution.

☐ 遇到根號類型, 應聯想到平方差公式: $(a+b)(a-b)=a^2-b^2$。

IgYe0g0pVgw

Example 5. Prove that $\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

Solution.

☐ 若有看到絕對值, 可以試著拆絕對值後考慮單邊極限。

UvarpOVMFOY

Theorem 6 (page 101)**.** *If $f(x)\le g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then*

$$\lim_{x\to a} f(x)\le \lim_{x\to a} g(x).$$

☐ 定理的條件若改成 $f(x)<g(x)$, 取極限後仍爲「小於等於」$\lim_{x\to a} f(x)\le \lim_{x\to a} g(x)$。

☐ 龜兎賽跑。

Theorem 7 (The Squeeze Theorem (夾擠定理; 三明治定理), page 101)**.** *If*

(1) $f(x)\le g(x)\le h(x)$ *when x is near a (except possibly at a), and*

(2) $\lim_{x\to a} f(x)=\lim_{x\to a} h(x)=L$,

then $\lim_{x\to a} g(x)=L$.

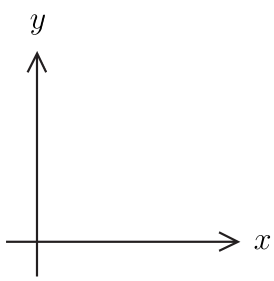

Figure 1: The Squeeze Theorem.

☐ 用簡單的函數控制複雜的函數。

Example 8. Show that $\lim_{x\to 0} x\sin\frac{1}{x} = 0$.

Solution.

Example 9. Show that $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

wXafFIZLbTA

Solution (page 192)**.** Assume first that x lies between 0 and $\frac{\pi}{2}$. Figure 2 shows a sector of a circle with center O, central angle x, and radius 1.

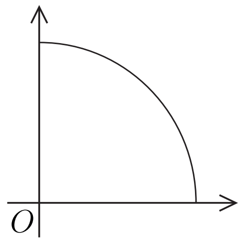

Figure 2: A sector of a circle with center O, central angle x, and radius 1.

Since "area of $\triangle OAB$" < "area of sector OAB" < "area of $\triangle OAC$", we have

If $-\frac{\pi}{2} < x < 0$, since $\sin x, x, \tan x$ are odd functions, we get $\tan x < x < \sin x < 0$, then

☐ 看待此極限的哲學: $\lim_{\bullet\to 0}\frac{\sin\bullet}{\bullet} = 1$, 只要 $\bullet$ 放一樣的東西即可, 所以 $\lim_{x\to 0}\frac{\sin 2x}{2x} =$ ___。

☐ 注意此極限是看 $\frac{\sin x}{x}$ 在「$x = 0$」附近的行爲。

Example 10. Find the limit $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

FDa-8AV9OL4

Solution. By the half-angle formula: $\sin^2\frac{x}{2} =$ __________, we get

Example 11. Find the following limits: $\lim_{x\to 0} \frac{x^3 \sin\left(\frac{1}{x}\right)}{\sin(x^2)}$.

Solution.

Solution 2.

VnFcngyxWXg

Example 12. Is there a number a such that

$$\lim_{x\to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

Solution.

zx4Mey6vKEg

Example 13 (夾擠定理的應用)**.**

(a) Show that: If $|f(x)| \leq |g(x)|$ and $\lim_{x\to a} |g(x)| = 0$, then $\lim_{x\to a} f(x) = 0$.

(b) Show that $\lim_{x\to 0} \sin x = 0$.

(c) Show that $\lim_{x\to a} \cos x = \cos a$ and $\lim_{x\to a} \sin x = \sin a$ for $a \in \mathbb{R}$.

Solution.

Question 14. How do we show that the limit $\lim_{x\to a} f(x)$ does not exist?

2.4 The Precise Definition of a Limit, page 104

Definition 1 (ε-δ language, page 106)**.** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the *limit of* $f(x)$ *as* x *approaches* a *is* L, and we write

1galLkYJM-s

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

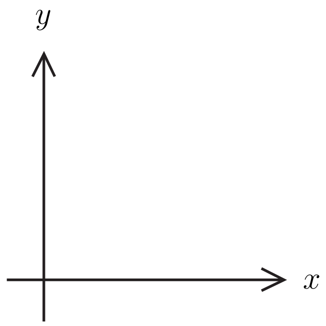

Figure 1: Limit of $f(x)$ as x approaches a is L.

☐ 極限的定義是“互動式”。

☐ δ 的找法在於“存在性”即可, 並不需要找到最佳的範圍。

☐ 邏輯符號: $\forall$ 代表 for all; $\exists$ 代表 exist。而 such that 數學上通常會簡記為 s.t.。

☐ 搭配邏輯符號, 極限的定義可寫成: ______________________________。

Example 2. Prove that $\lim_{x \to 1} (2x + 3) = 5$.

5TfkXX1np8k

Solution.

- Observation: We calculate $|(2x + 3) - 5| = |2x - 2| = 2|x - 1|$. We want to find $\delta > 0$ such that

$$\text{if} \quad 0 < |x - 1| < \delta, \quad \text{then} \quad 2|x - 1| < \varepsilon.$$
$$\text{That is,} \quad \text{if} \quad 0 < |x - 1| < \delta, \quad \text{then} \quad |x - 1| < \frac{\varepsilon}{2}.$$

  This suggests that we can choose $\delta = \frac{\varepsilon}{2}$. (or smaller)

- Proof:

OS-3Kd_kV78

Example 3. Prove that $\lim_{x \to 3} x^2 = 9$.

Solution.

- Observation: We calculate $|x^2 - 9| = |x+3||x-3| < \varepsilon$. We want to find $\delta > 0$ such that

$$\text{if} \quad 0 < |x-3| < \delta, \quad \text{then} \quad |x+3||x-3| < \varepsilon.$$

  Notice that if we can find a positive constant M such that $|x+3| < M$, then $|x+3||x-3| < M|x-3|$, and then we can make $M|x-3| < \varepsilon$ by taking $|x-3| < \frac{\varepsilon}{M} = \delta$.

  Since we are interested only in values of x that close to 3, it is reasonable to assume $|x-3| < 1$, then $|x+3| < 7$, so $M = 7$ is a choice.

- Proof:

☐ 在進行正式論述時只要寫 Proof 的那段論證即可。

5ZNKwxGoyyg

Example 4. Prove the Limit Sum Law: Suppose that the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.

Proof. For all $\varepsilon > 0$, since

☐

Question 5. How do we show that the limit $\lim_{x \to a} f(x)$ does not exist?

Solution. 證明極限不存在的其中一招是反證法: 假設極限存在, 記爲 $\lim_{x \to a} f(x) = L$, 然後要證明「任何的」$L \in \mathbb{R}$ 都會產生矛盾。那麼怎樣才有矛盾呢? 按極限定義, 必須證明 "there exists $\varepsilon > 0$, for all $\delta > 0$, there exists $0 < |x' - a| < \delta$ s.t. $|f(x') - L| \geq \varepsilon$"。

Solution 2. 因爲極限存在必唯一, 所以利用反證法: 若能設法找到「兩種數列」, 其極限值不同, 那麼就得到極限不存在。

Example 6. The *Dirichlet function* is defined by

PjVSOOsSDkE

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that $\lim_{x \to a} f(x)$ does not exist for every $a \in \mathbb{R}$.

Solution. Suppose $\lim_{x \to a} f(x) = L$. Notice that both rational numbers and irrational numbers are dense in real numbers.

<u>If $L \geq \frac{1}{2}$,</u>

<u>If $L < \frac{1}{2}$,</u>

☐ 我們無法正確畫出 Dirichlet 函數的圖形。

Example 7 (同學若有興趣可想一想這個例子)**.** The *Riemann function* is defined by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q}, (p, q) = 1 \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Then the limit of $f(x)$ exists as x approaches to any irrational number.

Definition 8 (Definition of left-hand limit, page 109)**.**

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a - \delta < x < a \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

Definition 9 (Definition of right-hand limit, page 109)**.**

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a < x < a + \delta \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

8IpPDC-7-QY

Definition 10 (page 112)**.** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = \infty$$

means that for every number M there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) > M.$$

☐ 無限大的意義: ______________________________。

Example 11. Prove that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Solution.

- Observation: Let M be a given positive number. We want to find a number $\delta > 0$ such that if $0 < |x| < \delta$, then $\frac{1}{x^2} > M$. Notice that

$$\frac{1}{x^2} > M \Leftrightarrow x^2 < \frac{1}{M} \Leftrightarrow |x| < \frac{1}{\sqrt{M}}.$$

  This suggests us to choose $\delta = \frac{1}{\sqrt{M}}$ (or smaller).

- Proof:

Definition 12 (page 112)**.** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = -\infty$$

means that for every number N there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) < N.$$

2.5 Continuity, page 114

TDC_ydWmW2U

Definition 1 (continuous at a point, page 114)**.** A function $f(x)$ is *continuous at* $x = a$ (在 $x = a$ 處連續) if

$$\lim_{x \to a} f(x) = f(a).$$

We say that $f(x)$ is *discontinuous at* $x = a$ (or $f(x)$ has a *discontinuity* at $x = a$) (在 $x = a$ 處不連續) if $f(x)$ is *not* continuous at a.

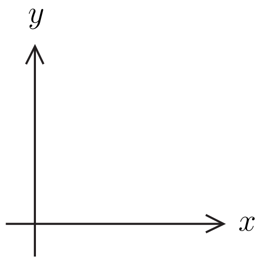

Figure 1: $f(x)$ is continuous at $x = a$.

☐ 函數 $f(x)$ 在 $x = a$ 處連續必須滿足以下三件事:

(1) $f(x)$ 在 $x = a$ 有定義, 即 $f(a)$ 存在。

(2) 極限 $\lim_{x \to a} f(x)$ 存在, 即左極限 $\lim_{x \to a^-} f(x)$ 與 $\lim_{x \to a^+} f(x)$ 存在且相等。

(3) $\lim_{x \to a} f(x) = f(a)$; 極限值等於函數值。

☐ 連續函數的另一個觀點:「極限」與「函數」可交換, 即 $\lim_{x \to a} f(x) = f(\lim_{x \to a} x) = f(a)$。

There are three types of discontinuity:

(1) *removable discontinuity* (可移不連續點): We can "redefine" the value of the function $f(x)$ at $x = a$ such that $f(x)$ is continuous at $x = a$.

(2) *infinite discontinuity* (無限不連續點).

(3) *jump discontinuity* (跳躍不連續點).

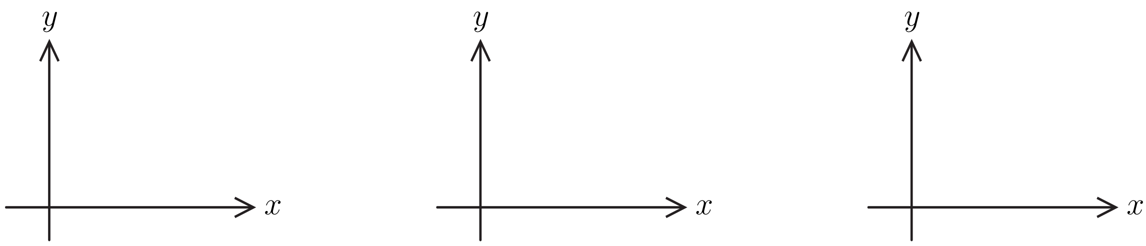

Figure 2: Three types of discontinuity.

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Definition 2 (continuous from the right (or left) (右連續與左連續), page 116)**.**

(a) A function $f(x)$ is *continuous from the right at* $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.

(b) A function $f(x)$ is *continuous from the left at* $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$.

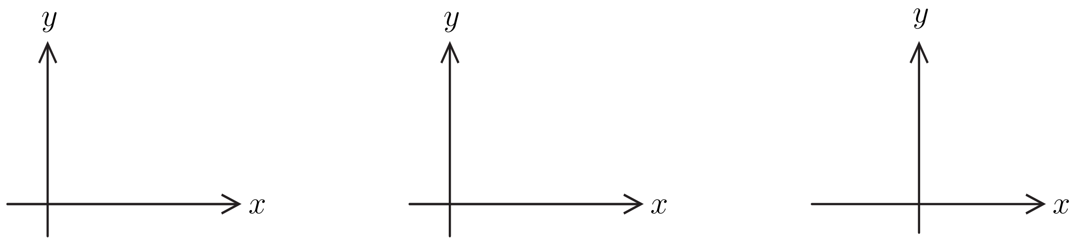

Figure 3: $f(x)$ is continuous (a) from the right; (b) from the left; (c) at endpoints.

Example 3. Discuss the continuity of the following functions:

$$f(x) = \frac{x^2 - x - 2}{x - 2}, \quad g(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2, \end{cases}, \quad h(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}.$$

Solution.

Example 4 (同學若有興趣可想一想這個例子)**.** The *Riemann function* is defined by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q}, (p, q) = 1 \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$

Then the Riemann function is continuous at irrational numbers.

7QUHBUSjD1I

Definition 5 (continuous on an interval, page 117)**.** A function $f(x)$ is *continuous on an interval* (在區間上連續) if it is continuous at every point in the interval. If $f(x)$ is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left.*

Theorem 6 (properties of continuous functions, page 117)**.** *If $f(x)$ and $g(x)$ are continuous at $x = a$, and c is a constant, then the following functions are also continuous $x = a$:*

(1) $(f \pm g)(x) = f(x) \pm g(x)$

(2) $cf(x)$, $cg(x)$

(3) $f(x)g(x)$

(4) $\frac{f(x)}{g(x)}$ *if* $g(a) \neq 0$.

Proof of (1). Since $f(x)$ and $g(x)$ are continuous at $x = a$, we have

Therefore

□

□ 連續函數經四則運算後仍爲連續函數 (除法要注意扣除分母的零點)。

Theorem 7 (page 120)**.** *The following type of functions are continuous at every number in their domains:*

polynomials	*rational functions*	*root functions*
trigonometric functions	*inverse trigonometric functions*	*exponential functions*
logarithmic functions		

Theorem 8 (page 120)**.** *If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$. In other words,*

wOGAe7ORjig

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b).$$

Example 9. Evaluate $\lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right)$

Solution.

Theorem 10 (page 121)**.** *If g is continuous at a and f is continuous at $g(a)$, then the composition function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.*

Proof. The function $g(x)$ is continuous at $x = a$ implies

□

□ 連續函數的合成函數也連續。

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Theorem 11 (The Intermediate Value Theorem, 中間值定理, page 122)**.** *Suppose that $f(x)$ is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.*

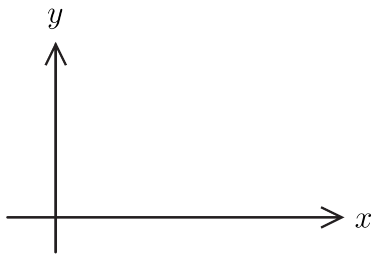

Figure 4: The Intermediate Value Theorem.

- ☐ “$f(x)$ is continuous” 是必要的。
- ☐ “closed” interval $[a, b]$ 是必要的。
- ☐ 交點個數並不唯一。

Applications of Intermediate Value Theorem

- ☐ 勘根定理
- ☐ 上山下山同時間
- ☐ 有限個點的平均和某點取值一樣
- ☐ 切蛋糕

7nfPoNfjTeY

Example 12. Suppose f is a continuous function on $[a, b]$ and $a \leq f(x) \leq b$ for all $x \in [a, b]$. Show that there exists $c \in [a, b]$ such that $f(c) = c$.

Solution.

2.6 Limits at Infinity: Horizontal Asymptotes, page 126

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Definition 1 (page 127–128)**.** Let f be a function defined on real numbers.

(a) $\lim_{x\to\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large (無窮遠處之極限).

(b) $\lim_{x\to-\infty} f(x) = L$ means that the value of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative. (負無窮遠處之極限)

☐ 極限 $\lim_{x\to\infty} f(x) = L$ 的另一種表達法爲 “$f(x) \to L$ as $x \to \infty$”。

☐ 極限 $\lim_{x\to-\infty} f(x) = L$ 的另一種表達法爲 “$f(x) \to L$ as $x \to -\infty$”。

Definition 2 (page 128)**.** The line $y = L$ is called a *horizontal asymptote* (水平漸近線) of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

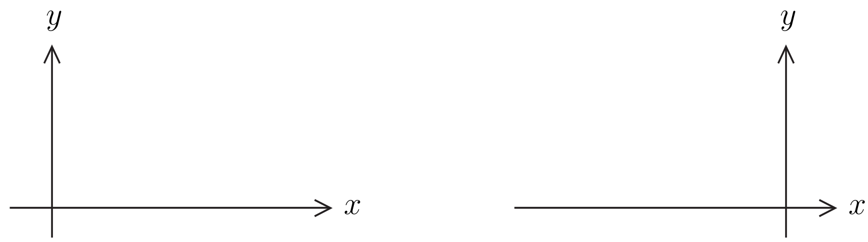

Figure 1: Limit $\lim_{x\to\infty} f(x) = L$, $\lim_{x\to-\infty} f(x) = L$, and horizontal asymptotes.

☐ 若一對一函數有垂直漸近線, 則其反函數有水平漸近線。例如: ________________。

Theorem 3 (page 129)**.**

(a) *If* $r > 0$ *is a rational number, then* $\lim_{x\to\infty} \frac{1}{x^r} = 0$.

(b) *If* $r > 0$ *is a rational number such that* x^r *is defined for all* x*, then* $\lim_{x\to-\infty} \frac{1}{x^r} = 0$.

Example 4. Evaluate $\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1}$.

Solution.

☐ 設 $P(x), Q(x)$ 爲兩多項式, 其領導係數分別爲 a_n 與 b_m, 則

$$\lim_{x\to\infty}\frac{P(x)}{Q(x)} = \begin{cases} & \text{if} \\ & \text{if} \\ & \text{if} \end{cases}$$

Wpr9zjJWC-Q

Example 5. Evaluate $\lim_{x\to\infty}(2^x + 3^x + 5^x)^{\frac{1}{x}}$.

Solution.

☐ 若有正數 $a_1, a_2, \ldots a_n$, 則 $\lim_{x\to\infty}((a_1)^x + (a_2)^x + \cdots + (a_n)^x)^{\frac{1}{x}} =$ ______________ 。

Example 6. Find the following limit:

$$\text{(a)} \lim_{x\to 0}\frac{\sin x}{x} \qquad \text{(b)} \lim_{x\to\infty}\frac{\sin x}{x} \qquad \text{(c)} \lim_{x\to 0} x\sin\frac{1}{x} \qquad \text{(d)} \lim_{x\to\infty} x\sin\frac{1}{x}.$$

Solution.

(a) $\lim_{x\to 0}\frac{\sin x}{x} =$ ___. (See section 2.3 **Example 9.**)

(b)

(c)

(d)

☐ 這個例題的心得是: ______________________________ 。

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Example 7. Evaluate the limit $\lim_{x\to\infty}\frac{3x^2+5}{5x+3}\sin\frac{2}{x}$.

Solution.

Example 8. Let $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \mathrm{e}$. Find $\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x$.

Solution.

Example 9. Find the limit $\lim_{x\to 0^+}(\cos\sqrt{x})^{\frac{1}{x}}$.

Solution.

☐ 學會變數變換與湊變數的能力。

Example 10. Let $f(x)=\sqrt{x^2+x}$. Compute the following limits:

(a) $A=\lim_{x\to\infty}\frac{f(x)}{x}$.

(b) $\lim_{x\to\infty}(f(x)-Ax)$.

q4XK7ih7RQs

Solution.

☐ 函數在無窮遠的行為, 有一類是存在「斜漸近線」。例如標準型的雙曲線 $x^2-y^2=1$。

fy4_4G-WAbw

Example 11. Suppose α, β are two constants and $\lim_{x\to-\infty}(\sqrt{x^2+3x+2}-\alpha x-\beta)=0$. Find α and β.

Solution.

☐ 計算 $x \to -\infty$ 的極限時要特別小心 x 是小於零的數。

☐ 害怕直接處理 $x \to -\infty$ 極限會出錯的話, 可以考慮用變數變換:

Infinite Limits at Infinity, Precise Definition

Definition 12 (page 134–137)**.**

(a) Let f be a function defined on some interval (a,∞). Then $\lim_{x\to\infty} f(x) = L$ means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x > N \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

(b) Let f be a function defined on some interval $(-\infty, a)$. Then $\lim_{x\to-\infty} f(x) = L$ means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x < N \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

(c) Let f be a function defined on some interval (a,∞). Then $\lim_{x\to\infty} f(x) = \infty$ means that for every positive number M there is a corresponding positive number N such that if $x > N$, then $f(x) > M$.

2.7 Derivatives and Rates of Change, page 140

Definition 1 (page 141)**.** The *tangent line* (切線) to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope (斜率)

-evUkuaTh74

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that this limit exists.

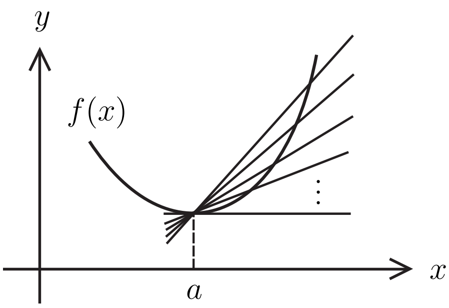

Figure 1: Tangent line is the limiting position of the secant line (割線).

☐ 切線斜率的定義是 ________________。

Example 2. Find an equation of the tangent line to the hyperbola $y = \frac{1}{x}$ at the point $(1, 1)$.

Solution. Let $f(x) = \frac{1}{x}$. Then the slope of the tangent at $(1, 1)$ is

Therefore, an equation of the tangent at the point $(1, 1)$ is ________________.

Definition 3 (page 143)**.** If $f(x)$ is the position function, then the *average velocity* (平均速度) is

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$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h},$$

and the *velocity* (or *instantaneous velocity*, 瞬時速度) $v(a)$ at time $t = a$ be the limit of these average velocities:

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

☐ The *speed* (速率) of the particle is the absolute value of the velocity $|v(a)|$.

Example 4. Suppose that a ball is dropped from the upper observation deck of Taipei 101, 508m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

Solution. Using the equation of motion $s = f(t) = 4.9t^2$, we have

(a) The velocity after 5 is __________.
(b) First we solve $4.9t_1^2 = 508$. This gives $t_1 = \sqrt{\frac{508}{4.9}}$. The velocity of the ball as it hits the ground is

KfYjrUE5RtA

Definition 5 (page 144)**.** The *derivative of a function* $f(x)$ *at a number* $x = a$ (函數 $f(x)$ 在 $x = a$ 的導數), denoted by $f'(a)$, is

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

If we use the point-slope form (點斜式) of the equation of a line, we can write an equation of the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$:

$$y - f(a) = f'(a)(x - a).$$

☐ 接下來的章節將討論如何計算各種函數的導數、導函數及其性質。

Example 6. Consider

$$f(x) = \begin{cases} x^\alpha \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

where α is a natural number. Determine whether $f'(0)$ exists.

Solution. By the definition of derivative, we have

☐ 這個例題很重要, 之後還有延伸的問題, 務必弄清楚。

Rates of Change

Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the *increment* (增加量) of x) is $\Delta x = x_2 - x_1$, and the corresponding change in y is $\Delta y = f(x_2) - f(x_1)$. The difference quotient

5-aJdX5hQhA

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the *average of the change of* y *with respect to* x (平均變化率) over the interval $[x_1, x_2]$. We say

$$\text{instantaneous rate of change } = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$ (瞬間變化率).

Examples of rates of change:

(1) Velocity of an object: the rate of change of displacement with respect to time.

(2) Marginal cost (邊際成本): the rate of change of production cost with respect to the number of items produced.

(3) Interest (in economics): the rate of change of the debt with respect to time.

(4) Power (in physics, 功率): the rate of change of work with respect to time.

(5) Rate of reaction (in chemistry): the rate of change in the concentration (濃度) of a reactant with respect to time.

(6) Rate of change of the population of a colony of bacteria with respect to time. (biology)

2.8 The Derivative as a Function, page 152

UtCufTsRzv8

Definition 1 (page 152)**.** The *derivative of* $f(x)$ ($f(x)$ 的導函數) is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

- ☐ 導函數 $f'(x)$ 的定義域是 $\{x \in \mathbb{R} | f'(x) \text{ exists}\}$。
- ☐ 左導函數記爲 $f'_-(x) = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$; 右導函數記爲 $f'_+(x) = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$。
- ☐ 導函數存在等價於左導函數與右導函數皆存在且相等。

Example 2. Let $f(x) = x^3$. Find $f'(x)$.

Solution. By definition,

Other Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = Df(x) = D_x f(x).$$

The symbols D and $\frac{\mathrm{d}}{\mathrm{d}x}$ are called *differentiation operators* (微分算子) because thy indicate the operation of *differentiation.*

We use the notation

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a} \quad \text{or} \quad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{x=a}$$

to indicate the value of a variable $\frac{\mathrm{d}y}{\mathrm{d}x}$ at a specific number a, which is a synonym for $f'(a)$.

HF9eAGoF_Bg

Definition 3 (page 155)**.** A function f is *differentiable at* a (在 $x = a$ 處可微分) if $f'(a)$ exists. It is *differentiable on an open interval* (a, b) (or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$) if it is differentiable at every number in the interval.

- ☐ 若函數只定義於 $[a, b]$, 則端點的導數就只要看 $\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$ 及 $\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h}$。

Theorem 4 (page 157)**.** *If f is differentiable at a, then f is continuous at a.*

Proof of **Theorem 4** is in the Appendix.

☐ Theorem 4 的逆敘述不對, 例如: $f(x) = |x|$。

☐ 數學上有一種函數是處處連續, 但處處不可微分。

How Can a Function Fail to Be Differentiable?

(1) corner or kink: the graph of f has no tangent at this point and f is not differentiable there.

(2) discontinuity: f is not continuous at a, then f is not differentiable at a.

(3) vertical tangent line: f is continuous at a and $\lim_{x\to a} |f'(x)| = \infty$.

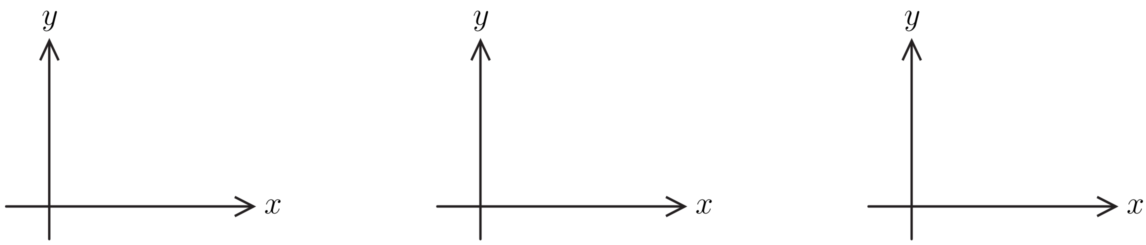

Figure 1: Three ways for $f(x)$ not to be differentiable at $x = a$.

Higher Derivatives

If f is differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the *second derivative* of f (二次導數). We write the second derivative of $y = f(x)$ as

b261rRqbKmA

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}.$$

☐ *acceleration* (加速度): 速度函數對時間的瞬間變化率。

The *third derivative* f''' (三次導數) is the derivative of the second derivative: $f''' = (f'')'$. If $y = f(x)$, then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) = \frac{\mathrm{d}^3 y}{\mathrm{d}x^3}.$$

In general, the *n-th derivative* ($n \geq 4$) of f is denoted by $f^{(n)}$. If $y = f(x)$, we write

$$y^n = f^{(n)}(x) = \frac{\mathrm{d}^n y}{\mathrm{d}x^n}.$$

☐ *jerk* (急動度、加加速度): 加速度的變化率。

Example 5. Suppose

$$f(x) = \begin{cases} \frac{1-\cos x}{\sin x} & x > 0 \\ ax + b & x \leq 0. \end{cases}$$

Find a and b such that f is continuous and differentiable at $x = 0$.

Solution.

☐ 想清楚函數在一個點「連續」、「可微分」的意義 (數學定義)。

mvhPYZQ-DuY

Example 6. Let $f(x) = x|x|$. Find $f'(x)$ and $f''(x)$.

Solution.

☐ 遇到分段定義的函數 (例如這個例子中的 $x = 0$ 處) 必須「用定義」小心處理。

☐ 割線斜率的極限 $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$ 和切線斜率的極限 $\lim\limits_{h\to 0} f'(x+h)$ 是兩個不同的概念。

☐ 記符號 $C^k(\mathbb{R})$ 爲所有 k 次求導後仍爲連續的函數所成的集合。

☐ 若 $f(x) \in C^1(\mathbb{R})$, 則有 $\lim\limits_{x\to a} f'(x) = f'(\lim\limits_{x\to a} x) = f'(a)$。

☐ 結論: $f(x) = x|x| \in C^1(\mathbb{R})$。

Appendix

Proof of Theorem 4. The goal is to show that $\lim\limits_{x\to a} f(x) = f(a)$.
For $x \neq a$, we have

3e2zttnxYy0

$$f(x) - f(a) = \frac{f(x)-f(a)}{x-a} \cdot (x-a),$$

so

$$\begin{aligned}\lim_{x\to a}(f(x)-f(a)) &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\cdot(x-a)\right) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a)\\ &= f'(a)\cdot 0 = 0.\end{aligned}$$

Hence

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}(f(x)-f(a)+f(a)) = \lim_{x\to a}(f(x)-f(a)) + \lim_{x\to a} f(a)\\ &= 0 + f(a) = f(a).\end{aligned}$$

☐