Chapter 1 Functions and Models

1.1 Four Ways to Represent a Function, page 10

Definition 1 (page 10)**.** A *function* (函數) f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$ in a set E.

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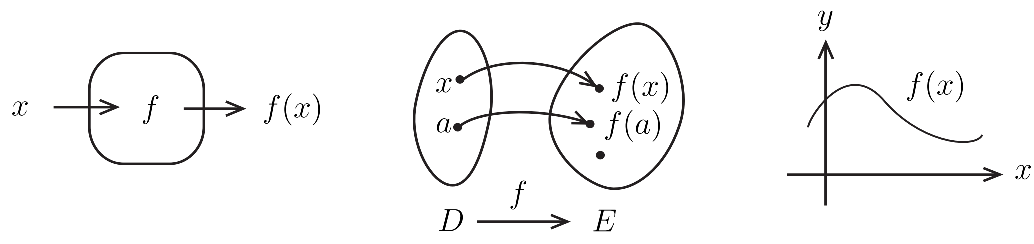

(a) (b) (c)

Figure 1: (a) machine diagram; (b) arrow diagram; (c) graph (圖形) of a function.

We usually consider functions for which the sets D and E are sets of real numbers $\mathbb{R}$.

☐ domain (定義域); codomain (對應域); range (值域).

☐ value of f at x (or "f of x").

☐ independent variable, dependent variable.

There are four possible ways to represent a function:

(1) verbally (by a description in words). 平常的溝通與交流

(2) numerically (by a table of values). 透過數據之觀察可發現一些現象

(3) visually (by a graph). 視覺引導通常會帶來深刻印象, 但有時圖形無法如實呈現

(4) algebraically (by an explicit formula). 數學上的嚴謹性充足, 但有時候不直覺

Example 2. Find the domain of the following function:

(1) $f(x) = \frac{x^2}{1+x}$. $D = \{x \in \mathbb{R} \mid \qquad\}$.

(2) $f(x) = (x-2)\sqrt{\frac{1+x}{1-x}}$. $D = \{x \in \mathbb{R} \mid \qquad\}$.

(3) $f(x) = \log(x+2) + \log(x-2)$. $D = \{x \in \mathbb{R} \mid \qquad\}$.

(4) $f(x) = \tan x$. $D = \{x \in \mathbb{R} \mid \qquad\}$.

E48F275yPbE

Vertical Line Test (page 15)**.** *A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.*

☐ 判斷一條曲線是否可以表示成函數的圖形, 幾何上使用「鉛直線法」。

Example 3. Give examples that one curve is the graph of a function and one curve is not the graph of a function.

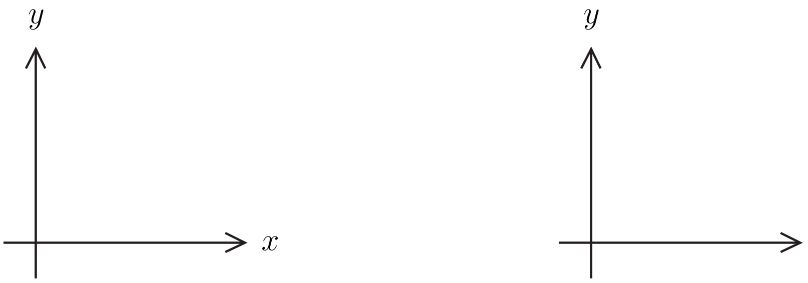

Figure 2: Left curve is a graph of a function; Right curve is not a graph of a function.

Example 4 (page 16)**.** The *absolute value* (絕對值) of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line. The graph of the absolute value function is

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

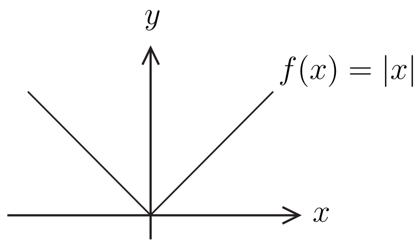

Figure 3: The graph of the absolute value function.

Example 5. Sketch the graph of the *Heaviside function* $H(x)$, which is defined by

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

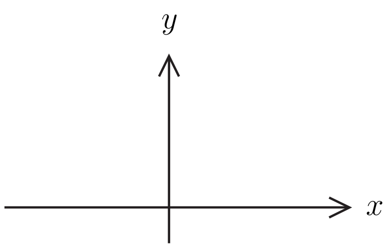

Figure 4: The Heaviside function.

Example 6. The graph of the *sign function* sgn(x) (符號函數), which is defined by

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

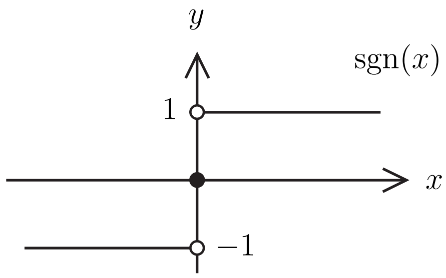

Figure 5: The sign function sgn(x).

☐ 這個函數稱爲符號函數的原因是: ________________。

Definition 7 (Odd function and even function, page 17–18)**.**

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(a) If a function f satisfies $f(-x) = -f(x)$ for every number x in the domain, then f is called an *odd function* (奇函數).

(b) If a function f satisfies $f(-x) = f(x)$ for every number x in the domain, then f is called an *even function* (偶函數).

☐ 奇函數的例子: ________________。

☐ 偶函數的例子: ________________。

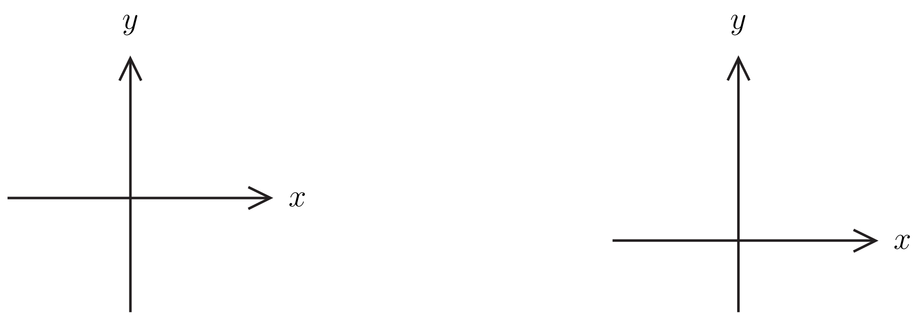

Figure 6: Left: an odd function; Right: an even function.

☐ 所有奇函數圖形必對稱於 ________。

☐ 所有偶函數圖形必對稱於 ________。

-oadKznW0pE

Example 8. Any function defined on $\mathbb{R}$ can be uniquely decomposed as the sum of an odd function and an even function.

Proof. Define two functions

$$g(x) = \frac{f(x) - f(-x)}{2} \quad \text{and} \quad h(x) = \frac{f(x) + f(-x)}{2}.$$

We will show that

- $g(x)$ is an odd function:

- $h(x)$ is an even function:

- $f(x) = g(x) + h(x)$:

□

□ 上述的證明只說明了「存在性」, 而「唯一性」的部份必須重新論述。

XIYWqfZBh8M

Definition 9 (Increasing and decreasing functions, page 19)**.**

(a) A function $f(x)$ is called *increasing* (遞增) on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

(b) A function $f(x)$ is called *decreasing* (遞減) on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.

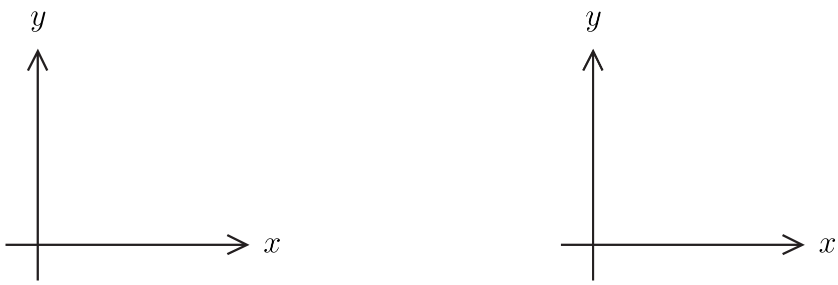

Figure 7: Increasing function and decreasing function.

□ 教科書與微積分課用 increasing 及 decreasing 等詞彙時, 函數值的比較都是「不等號」。

□ 有些書或文獻會用 “strictly” 或 “monotone” increasing (decreasing) 強調不等號。

1.2 Mathematical Models: A Catalog of Essential Functions, page 23

Mathematical models

Why do we learn mathematics? One reason is that mathematics can help us solve problems.

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$$\begin{array}{ccc} \text{Real-world problem} & \longrightarrow & \text{Mathematical model} \\ & & \downarrow \\ \text{Real-world predictions} & \longleftarrow & \text{Mathematical conclusions} \end{array}$$

Essential functions

Definition 1 (page 27)**.** A function P is called *polynomial* (多項式) if

DwYfp5nIoyM

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where n is a nonnegative integer and the numbers $a_0, a_1, \ldots, a_n$ are constants called the *coefficients* (係數) of the polynomial. If the leading coefficient $a_n \neq 0$, then the *degree* (次數) of the polynomial is n.

(1) A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a *linear function.* (中學數學學會畫線性函數)

(2) A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and so it is a *quadratic function.* (中學數學學會畫二次函數)

(3) A polynomial of degree 3 is of the form $P(x) = ax^3 + bx^2 + cx + d$ and so it is a *cubic function.* (學完微積分可以知道所有的三次多項式)

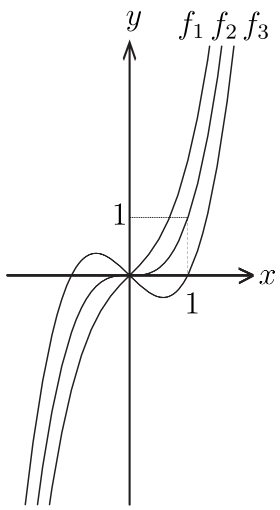

Figure 1: Some cubic functions: $f_1(x) = x^3 + x, f_2(x) = x^3, f_3(x) = x^3 - x$.

Desmos Calculator (數學繪圖軟體)

J2ewDdaSRmo

Desmos Calculator is a free, online, mathematical software. The link is:

`https://www.desmos.com/calculator`

It is easy to use Desmos Calculator to plot the graphs of functions, equations and inequalities. More advanced features are well-designed such as polar function graphing. Users can use their Gmail account to login and save the graphs.

In calculus class, we will often use Desmos Calculator to get familiar with many concepts such as derivatives, tangent lines, polar curves, Taylor expansions, etc. Furthermore, many important graph properties such as symmetry, shift, stretching, reflection, rotation, can be easily and clearly presented by Desmos Calculator.

Definition 2 (page 29)**.** A function of the form $f(x) = x^a$, where a is a constant, is called a *power function* (冪函數).

We usually consider several cases.

(a) $a = n$, where n is a positive integer *polynomial with one term* (多項式).

(b) $a = \frac{1}{n}$, where n is a positive integer. *root function* (根式函數).

(c) $a = -1$. *reciprocal function* (倒數函數).

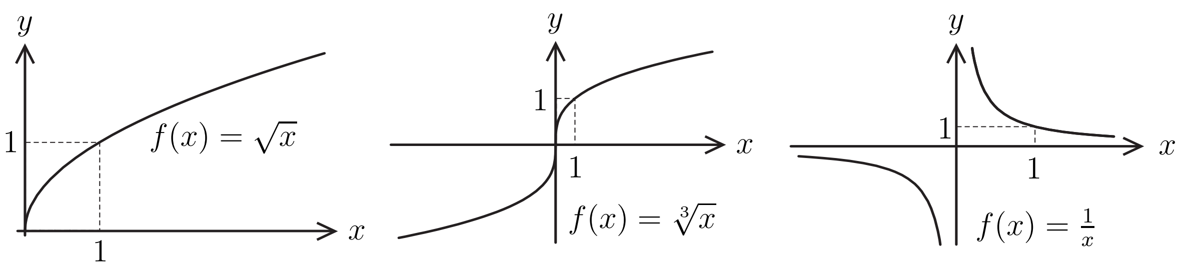

Figure 2: Graphs of root function and reciprocal function.

Definition 3 (page 30)**.** A *rational function* (有理函數) is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)},$$

where P and Q are polynomials. The domain consists of all values of x such that $Q(x) \neq 0$.

Definition 4 (page 30)**.** A function f is called an *algebraic function* (代數函數) if it can be construct using algebraic operations (addition, subtraction, multiplication, division, and roots) starting with polynomials.

Example 5. The mass of a particle with velocity v is $m = f(v) = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, where m_0 is the rest mass of the particle; $c = 3 \times 10^5$ km/s is the speed of light in a vacuum.

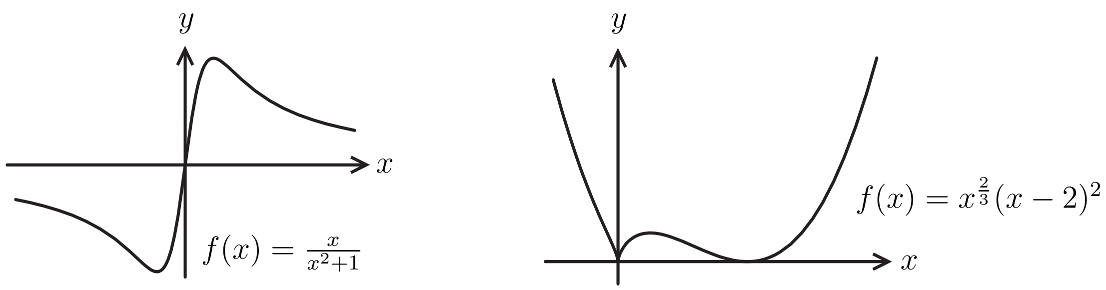

Figure 3: Graphs of rational function and algebraic function.

Definition 6 (page A26)**.** Let $P(x, y)$ by any point on the terminal side of θ and let r be the distance $|OP|$. Then we define *trigonometric functions* (三角函數) as:

s1rSBpGkVuE

$$\sin\theta = \frac{y}{r} \quad \cos\theta = \frac{x}{r} \quad \tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y} \quad \sec\theta = \frac{r}{x} \quad \csc\theta = \frac{r}{y}.$$

Angles can be measured in degrees or in radians (abbreviated as rad). The angle given by a complete revolution contain 360°, which is the same as 2π rad.

There are a lot of *trigonometric identities* (三角恆等式):

2jGcpkNkR-w

☐ $\sin\theta\csc\theta = 1, \quad \cos\theta\sec\theta = 1, \quad \tan\theta\cot\theta = 1.$

☐ $\cos\theta\tan\theta = \sin\theta, \quad \sin\theta\cot\theta = \cos\theta, \quad \sin\theta\sec\theta = \tan\theta,$
$\cos\theta\csc\theta = \cot\theta, \quad \tan\theta\csc\theta = \sec\theta, \quad \cot\theta\sec\theta = \csc\theta.$

☐ $\sin^2\theta + \cos^2\theta = 1, \quad \tan^2\theta + 1 = \sec^2\theta, \quad 1 + \cot^2\theta = \csc^2\theta.$

☐ $\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta, \quad \tan(-\theta) = -\tan\theta,$
$\cot(-\theta) = -\cot\theta, \quad \sec(-\theta) = \sec\theta, \quad \csc(-\theta) = -\csc\theta.$

☐ $\sin(\theta + 2\pi) = \sin\theta, \quad \cos(\theta + 2\pi) = \cos\theta, \quad \tan(\theta + \pi) = \tan\theta,$
$\cot(\theta + \pi) = \cot\theta, \quad \sec(\theta + 2\pi) = \sec\theta, \quad \csc(\theta + 2\pi) = \csc\theta.$

☐ $\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y,$
$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y,$
$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}.$

☐ $\sin 2x = 2\sin x\cos x,$
$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x.$

☐ $\cos^2 x = \frac{1+\cos 2x}{2},$
$\sin^2 x = \frac{1-\cos 2x}{2}.$

☐ $2\sin x\cos y = \sin(x+y) + \sin(x-y)$
$2\cos x\cos y = \cos(x+y) + \cos(x-y)$
$2\sin x\sin y = -\cos(x+y) + \cos(x-y)$

Definition 7. A function $f(x)$ is called *bounded on an interval* I (有界函數) if there exists a constant M such that $|f(x)| \leq M$ for all $x \in I$.

Example 8. $\sin x, \cos x$ are bounded functions on $\mathbb{R}$. $\sin\left(\frac{1}{x}\right)$ is bounded on $x \neq 0$. $|x|$ is not bounded function on $\mathbb{R}$.

Definition 9. A function $f(x)$ is called *periodic* with period T (nonzero constant) (周期函數) if $f(x+T) = f(x)$ for x is defined.

Example 10. $\sin x, \cos x, \sec x$ and $\csc x$ are periodic functions with period 2π; $\tan x$ and $\cot x$ are periodic functions with period π.

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Definition 11 (page 32)**.** The *exponential function* (指數函數) are the functions of the form $f(x) = a^x$, where the basis a is a positive constant.

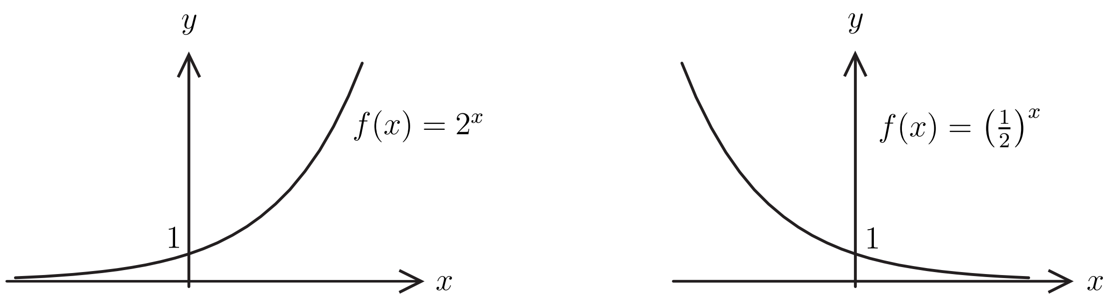

Figure 4: Exponential functions.

Definition 12 (page 32)**.** The *logarithmic function* (對數函數) $f(x) = \log_a x$, where the base a ia a positive constant, are the inverse functions of the exponential functions.

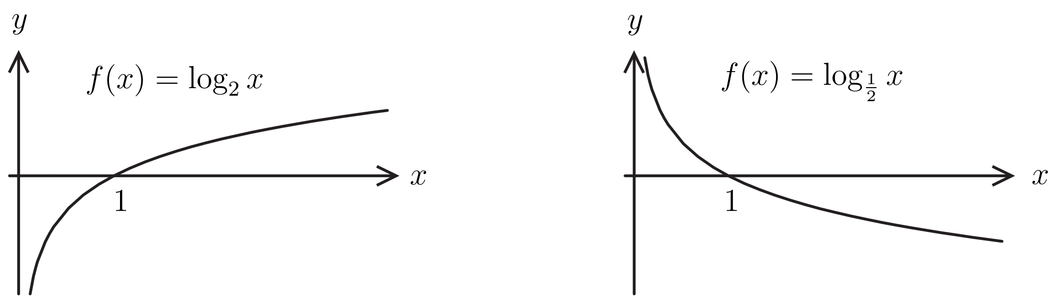

Figure 5: Logarithmic functions.

1.3 New Functions from Old Functions, page 36

Given a function $f(x)$, we will discuss the new function

$$g(x) = af(bx + c) + d$$

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for constants a, b, c, d and the effects of these constants.

- ☐ The effect of a is vertical dilation (上下伸縮, 對 x-軸).
- ☐ The effect of b is horizontal dilation (左右伸縮, 對 y-軸).
- ☐ The effect of c is horizontal shift (左右平移).
- ☐ The effect of d is vertical shift (上下平移).

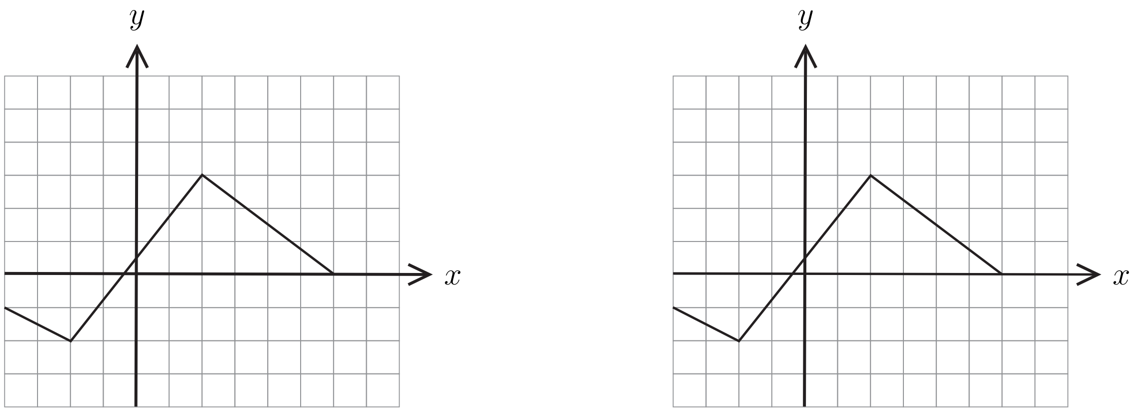

Figure 1: The effect of $d = 3$ and $a = 2$.

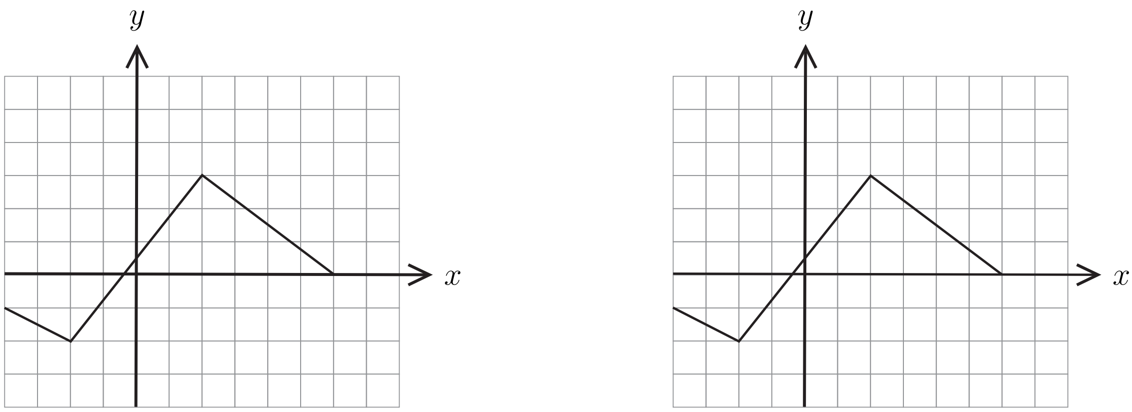

Figure 2: The effect of $c = -2$ and $b = 2$.

- ☐ $0 < b < 1$: 橫向膨脹。$b > 1$: 橫向壓縮。$c < 0$: 向右平移。$c > 0$: 向左平移。
- ☐ 在函數裡面的係數效應要注意 (有“相反”的意味)。
- ☐ 從 $f(x)$ 要畫出 $af(bx + c) + d$, 處理順序爲 $c \to b \to a \to d$ 或 $a \to d \to c \to b$。

Example 1. Starting from $f(x) = x^2$, plot the function $f_4(x) = \frac{1}{3}(2x+5)^2 - 1$.

Solution.

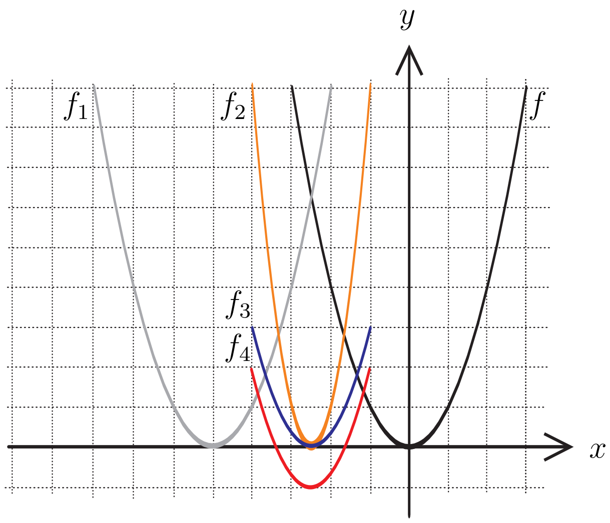

Figure 3: $f(x) = x^2, f_1(x) = (x+5)^2, f_2(x) = (2x+5)^2, f_3(x) = \frac{1}{3}(2x+5)^2$, and $f_4(x) = \frac{1}{3}(2x+5)^2 - 1$.

d9sGY-4gITs

Definition 2 (page 41)**.** Given two functions f and g, the *composite function* $f \circ g$ (also called the *composition* of f and g) (合成函數) is defined by

$$(f \circ g)(x) = f(g(x)).$$

- ☐ 記號 $f \circ g$ 是先作用 g, 再作用 f。
- ☐ 一般而言, $f \circ g \neq g \circ f$。

Example 3 (Composition of a function and the absolute value function)**.** Discuss the relations between $f(x)$, $|f(x)|$, and $f(|x|)$.

Solution.

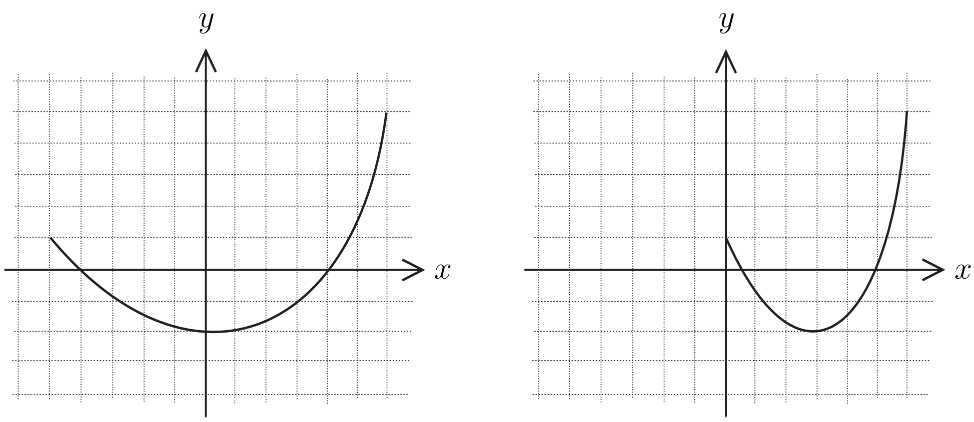

Figure 4: Left: Given $f(x)$, plot $|f(x)|$. Right: Given $f(x)$, plot $f(|x|)$.

Example 4. Given $f(x)$, compare the process $c \to b \to a \to d$ with $b \to c \to a \to d$.

2wBVaqvFTZE

Solution. Given a function $f(x)$, when we consider steps $c \to b \to a \to d$, it would be

$$f_1(x) =$$
$$f_2(x) =$$
$$f_3(x) =$$
$$f_4(x) =$$

When we consider steps $b \to c \to a \to d$, it would be

$$\tilde{f}_1(x) =$$
$$\tilde{f}_2(x) =$$
$$\tilde{f}_3(x) =$$
$$\tilde{f}_4(x) =$$

☐ 先後順序不同 ($b \to c$ 與 $c \to b$, 還有 $a \to d$ 與 $d \to a$) 會導致不同的效果。

☐ 新函數 $af(bx + c) + d$ 是由原函數 $f(x)$ 與一系列函數作合成而得。

1.4 Exponential Functions, page 51

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Definition 1 (page 45)**.** An *exponential function* (指數函數) is a function of the form

$$f(x) = a^x,$$

where a is a positive constant.

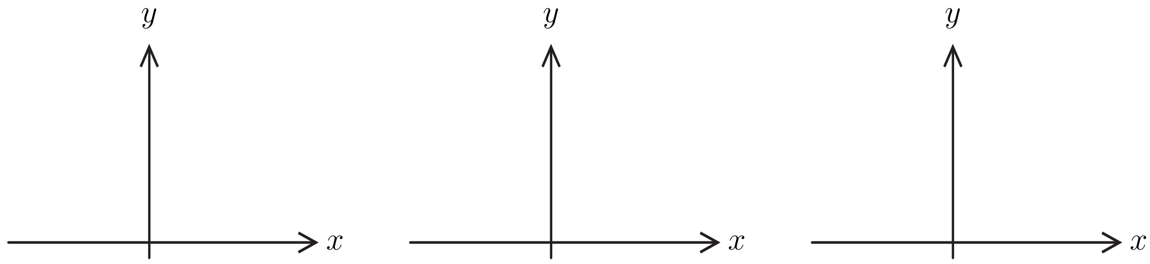

Figure 1: Three types of exponential functions $y = a^x$: $0 < a < 1$, $a = 1$, $a > 1$.

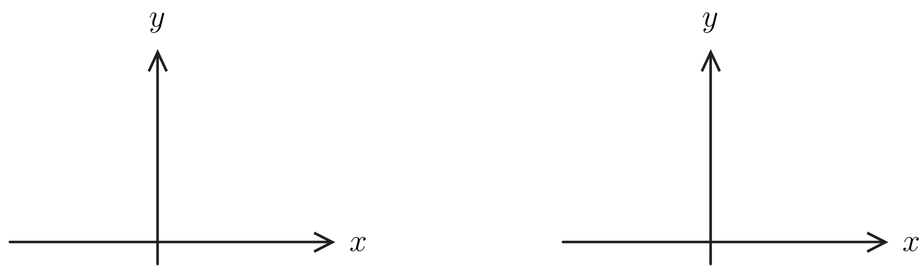

Figure 2: Compare $y = a^x$ and $y = b^x$. Left: $0 < a < b < 1$; Right: $1 < a < b$.

Laws of Exponents (page 47)**.** *If a and b are positive numbers and x and y are any real numbers, then*

(1) $a^{x+y} = a^x a^y$.

(2) $a^{x-y} = \frac{a^x}{a^y}$.

(3) $(a^x)^y = a^{xy}$.

(4) $(ab)^x = a^x b^x$.

Example 2. The meaning of 2^{3^2} is __________.

Applications of exponential functions

☐ 大腸桿菌的生菌數: 大腸桿菌進行無性生殖, 約 20 分鐘分裂一次。

☐ 人口增長模型: 自 18 世紀開始馬爾薩斯便以此進行人口與社會狀況的研究。

☐ 放射性元素的半衰期: 判定化石或地層的年代。

☐ 銀行複利問題。

The number e

Example 3. 某人拿 1 元去「佛心銀行」存款, 只見銀行櫃姐口沫橫飛地推銷最新優儲方案: 利率 100%, 並以複利計算, 可選擇年複利、季複利、月複利、日複利、時複利、分複利、秒複利。

tdaly0-xhLg

(1) 哪種方案的本利和最多?

(2)「佛心銀行」有沒有可能佛心到以某一種複利計算之下, 一年後的本利和任意大?

Solution.

(1) 複利公式爲:

$$\text{本利和} = \text{本金} \times \left(1 + \frac{\text{利率}}{\text{期數}}\right)^{\text{期數}},$$

所以我們可以得到各種複利方式的本利和, 記 n 表複利的期數, 於是

複利	n	本利和
年複利	1	2
季複利	4	2.44140625
月複利	12	2.61303529
日複利	365	2.71456748
時複利	8760	2.71812669
分複利	525600	2.71827924
秒複利	31536000	2.71828178

所以秒複利的方案獲利最多。

(2) 由二項式定理知:

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n &= 1 + C_1^n\left(\frac{1}{n}\right) + C_2^n\left(\frac{1}{n}\right)^2 + C_3^n\left(\frac{1}{n}\right)^3 + \cdots + C_n^n\left(\frac{1}{n}\right)^n \\
&\le 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\
&\le 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)\cdot n} \\
&\le 1 + 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\
&\le 1 + 1 + 1 - \frac{1}{n} < 3,
\end{aligned}$$

所以不會有一種複利方案讓本利和無限制增大。

瑞士數學家 Leonhard Euler 於 1727 年發現當 n 愈來愈大的時候, $\left(1+\frac{1}{n}\right)^n$ 不可能無止盡地增大, 並證明它有一個「極限」(至於「極限」到底是什麼意思, 會在第 2 章的時候會詳細說明)。他首先以符號 e 代表 $\left(1+\frac{1}{n}\right)^n$ 當 n 愈來愈大的極限值。這個數字對往後微積分以及近代數學的發展相當重要。

1.5 Inverse Functions and Logarithms, page 55

P0Gv3OKNdK4

Definition 1 (page 55)**.** A function f is called a *one-to-one function* (一對一函數) if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2.$$

☐ 一對一函數的等價敘述是: 若 $f(x_1) = f(x_2)$, 則 $x_1 = x_2$.

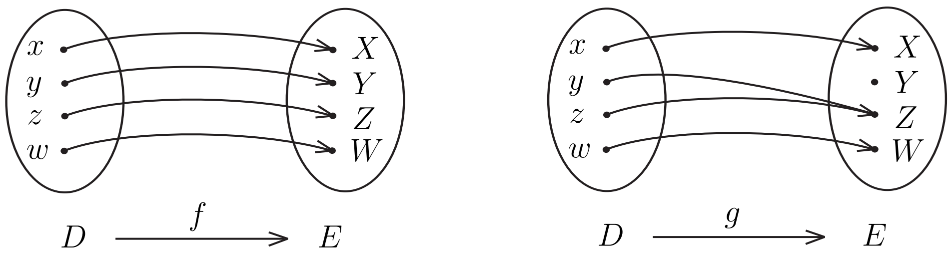

Figure 1: f is a one-to-one function; g is not a one-to-one function.

Horizontal Line Test (page 56)**.** *A function is one-to-one if and only if no horizontal line intersects its graph more than once.*

☐ 判斷一個函數圖形是否爲一對一函數, 幾何上使用「水平線法」。

Example 2. Plot the graphs of a one-to-one and not a one-to-one function.

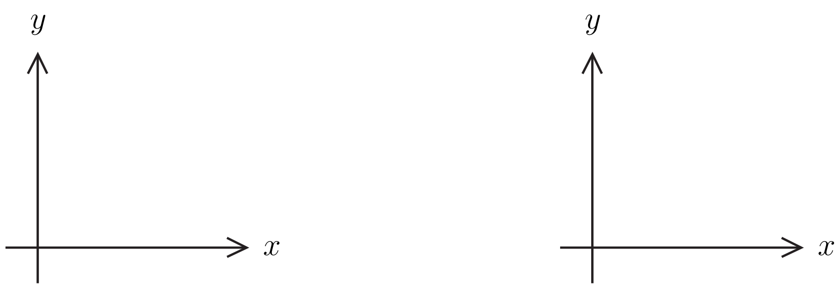

Figure 2: Left: a one-to-one function; Right: not a one-to-one function.

Definition 3 (page 56)**.** Let f be a one-to-one function with domain D and range E. Its *inverse function* (反函數) f^{-1} with domain E and range D is defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y \quad \text{for any } y \text{ in } E.$$

☐ $f^{-1}(x)$ 是逆運算, 不同於函數的倒數 $\frac{1}{f(x)} = (f(x))^{-1}$。

☐ 數學上常用 x 爲自變數, y 爲應變數, 所以反函數會寫成 $y = f^{-1}(x) \Leftrightarrow f(y) = x$。

☐ 函數與反函數有消去律: $f^{-1}(f(x)) = x \ \ \forall x \in D$; $f(f^{-1}(x)) = x \ \ \forall x \in E$。

Question 4. How do we find the inverse function of a one-to-one function $y = f(x)$?

(1) Solve this equation for x in terms of y (if possible).

(2) Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Example 5. Find a formula for the inverse of the function $f(x) = x^2 - x, x \geq \frac{1}{2}$.

3gyf0o9sUgY

Solution.

☐ 所有函數圖形 $y = f(x)$ 與其反函數圖形 $y = f^{-1}(x)$ 必對稱於 ______。

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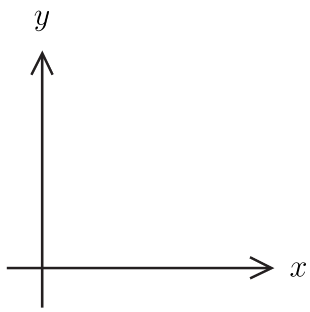

Figure 3: Symmetry of a function and its inverse function.

☐ 任何遞增 (遞減) 函數必存在反函數。(從 _____ 理解或從 __________ 看之。)

Example 6. Plot the graph $f(x) = \frac{1}{\sqrt{x}} = \frac{1}{x^{\frac{1}{2}}}$ and $g(x) = \frac{1}{x^2}, x > 0$.

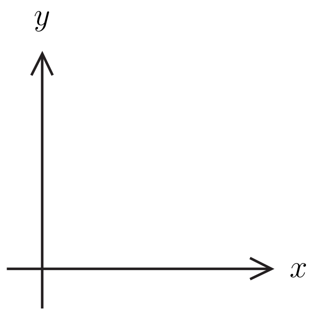

Figure 4: The graphs of $f(x) = \frac{1}{\sqrt{x}}$ and $g(x) = \frac{1}{x^2}, x > 0$.

Logarithmic Functions

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Definition 7 (page 59)**.** If $a > 0$ and $a \neq 1$, the exponential function $f(x) = a^x$ is either increasing or decreasing. It therefore has an inverse function $f^{-1}(x)$, which is called the *logarithmic function with base a* (以 a 爲底的對數函數) and is denoted by $\log_a x$.

Since $y = f^{-1}(x) \Leftrightarrow f(y) = x$, we have $y = \log_a x \Leftrightarrow a^y = x$.

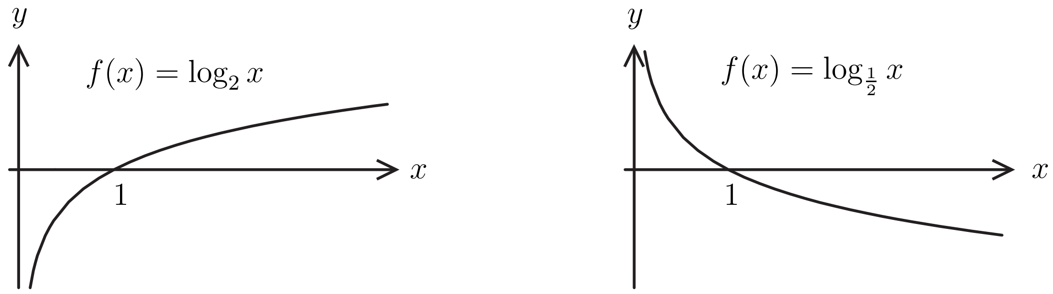

Figure 5: Logarithmic functions.

☐ 對所有 $x \in \mathbb{R}$, $\log_a(a^x) = x$ (由函數與反函數的消去律可看出)。

☐ 對所有 $x > 0$, $a^{\log_a x} = x$ (由函數與反函數的消去律可看出)。

Laws of Logarithms (page 59)**.** *If x and y are positive numbers, then*

(1) $\log_a(xy) = \log_a x + \log_a y$.

(2) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.

(3) $\log_a(x^r) = r \log_a x$ (*where r is any real number*).

Definition 8 (page 60)**.** The logarithm with base $\mathrm{e} \approx 2.718281828\ldots$ is called the *natural logarithm* (自然對數) and has a special notation:

$$\log_{\mathrm{e}} x = \ln x.$$

☐ $\ln x = y \Leftrightarrow \mathrm{e}^y = x$.

☐ $\ln(\mathrm{e}^x) = x$ for all $x \in \mathbb{R}$.

☐ $\mathrm{e}^{\ln x} = x$ for all $x > 0$.

☐ $\ln \mathrm{e} = 1$.

Property 9 (Change of base formula, page 62)**.** *For any positive number a ($a \neq 1$),*

$$\log_a x = \frac{\ln x}{\ln a}.$$

Inverse Trigonometric Functions

The sine function $f(x) = \sin x$ is not one-to-one, but the function $f(x) = \sin x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ is one-to-one.

NvaPqnxzI_A

Definition 10 (page 63)**.** The inverse function of this restricted sine function exists and is denoted by $\sin^{-1} x$ or $\arcsin x$. It is called the *inverse sine function* or the *arcsine function* (反正弦函數).

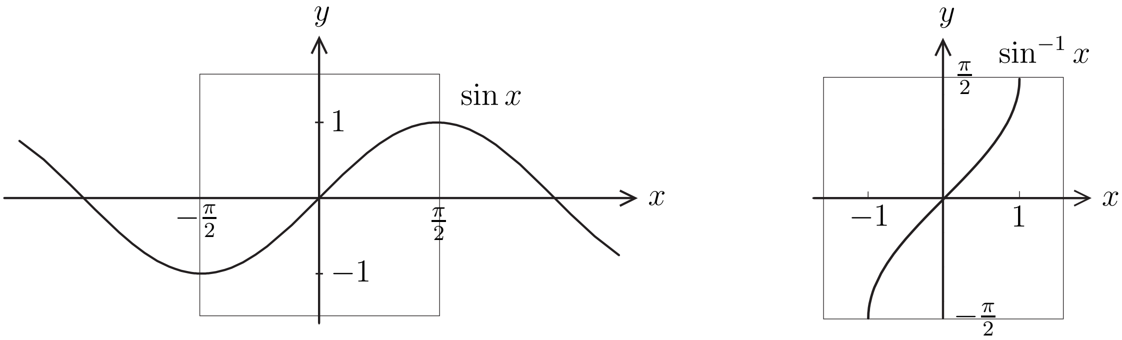

Figure 6: We first restrict $\sin x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then define its inverse function $\sin^{-1} x$.

☐ 特別注意 $\sin^{-1} x \neq \frac{1}{\sin x} = (\sin x)^{-1} = \csc x \neq \sin(x^{-1}) = \sin\left(\frac{1}{x}\right)$。

☐ 消去律得知 $\sin^{-1}(\sin x) = x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ 及 $\sin(\sin^{-1} x) = x$ for $-1 \leq x \leq 1$。

Example 11. Evaluate (a) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ and (b) $\tan\left(\arcsin \frac{2}{3}\right)$.

Solution.

Definition 12 (反餘弦、反正切、反餘切、反正割、反餘割函數, page 66)**.**

	Restriction	Inverse function	Notation
$\cos x$	$0 \leq x \leq \pi$	*inverse cosine function*	$\cos^{-1} x$ or $\arccos x$
$\tan x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$	*inverse tangent function*	$\tan^{-1} x$ or $\arctan x$
$\cot x$	$0 < x < \pi$	*inverse cotangent function*	$\cot^{-1} x$ or $\operatorname{arccot} x$
$\sec x$	$0 \leq x < \frac{\pi}{2}$ or $\pi \leq x < \frac{3\pi}{2}$	*inverse secant function*	$\sec^{-1} x$ or $\operatorname{arcsec} x$
$\csc x$	$0 < x \leq \frac{\pi}{2}$ or $\pi < x \leq \frac{3\pi}{2}$	*inverse cosecant function*	$\csc^{-1} x$ or $\operatorname{arccsc} x$.

☐ 關於 $\operatorname{arcsec} x$ 與 $\operatorname{arccsc} x$ 的值域, 並沒有統一的限制範圍。

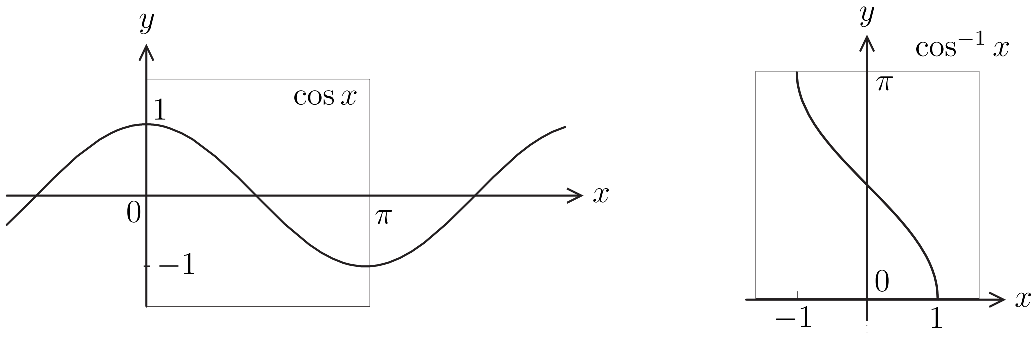

Figure 7: We first restrict $\cos x$ on $x \in [0, \pi]$, then define its inverse function $\cos^{-1} x$.

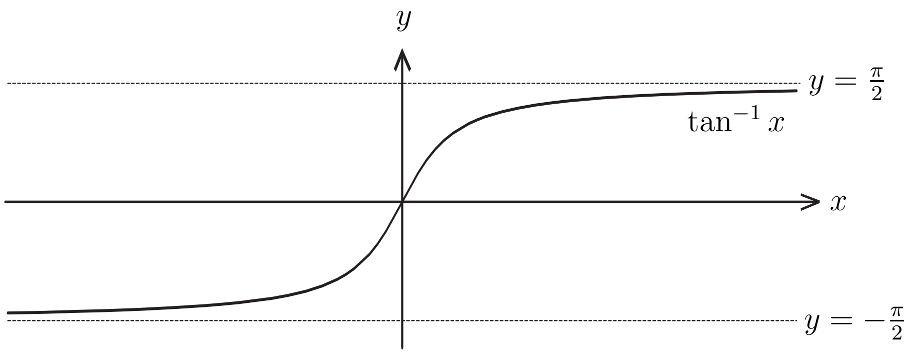

Figure 8: We first restrict $\tan x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$, then define its inverse function $\tan^{-1} x$.

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Example 13. Simplify the expression $\sin(\tan^{-1} x)$.

Solution.